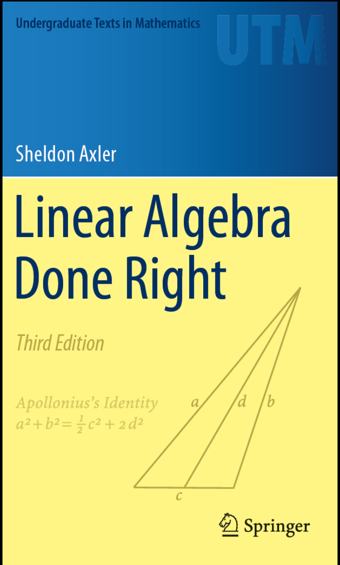
# Definition of Vector Space



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**Definition: *addition, scalar multiplication***

- An *addition* on a set $V$ is a function that assigns an element $u + w \in V$ to each pair of elements $u, w \in V$.
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Example: Suppose $V$ is the set of real-valued functions on the interval $[0, 1]$. For $f, g \in V$ and $\lambda \in \mathbf{R}$, define $f + g$ and $\lambda f$ by

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Thus $f + g \in V$ and $\lambda f \in V$.

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Addition and scalar multiplication on $\mathbf{F}^\infty$ are defined as expected:

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots),$$
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With these definitions of addition and scalar multiplication, $\mathbf{F}^S$ becomes a vector space.

# Our First Theorem

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which can be rewritten as

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as desired. ■

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An element of $V$ is not necessarily of the form $(x_1, \ldots, x_n)$.

*Linear Algebra Done Right*, by Sheldon Axler