Definition: **matrix of an operator, $\mathcal{M}(T)$**

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of $V$. The *matrix of $T$* with respect to this basis is the $n$-by-$n$ matrix

\[
\mathcal{M}(T) = \begin{pmatrix}
A_{1,1} & \cdots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n}
\end{pmatrix}
\]

whose entries $A_{j,k}$ are defined by

\[Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.\]
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If the basis is not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used.
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Suppose \( T \in \mathcal{L}(V) \), then
- \( \mathcal{M}(T) \) is computed using just one basis;
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Suppose \( T \in \mathcal{L}(V) \), then

- \( \mathcal{M}(T) \) is computed using just one basis;
- \( \mathcal{M}(T) \) is a *square* matrix.
Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of $V$. The *matrix of $T$ with respect to this basis* is the $n$-by-$n$ matrix

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Suppose $T \in \mathcal{L}(V)$, then

- $M(T)$ is computed using just one basis;
- $M(T)$ is a **square** matrix.

Example: Define $T \in \mathcal{L}(\mathbb{R}^3)$ by

\[
T(x, y, z) = (2x+y, 5y+3z, 8z).
\]

Then

\[
M(T) = \begin{pmatrix}
2 & 1 & 0 \\
0 & 5 & 3 \\
0 & 0 & 8
\end{pmatrix}
\]

with respect to the standard basis of $\mathbb{R}^3$. 

---

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If the basis is not clear from the context, then the notation $M(T, (v_1, \ldots, v_n))$ is used.
Upper-Triangular Matrices

Definition: *diagonal of matrix*

The *diagonal* of a square matrix consists of the entries along the line from the upper left corner to the bottom right corner.

Example:

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\begin{pmatrix}
2 & 1 & 0 \\
0 & 5 & 3 \\
0 & 0 & 8
\end{pmatrix}
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Conditions for upper-triangular matrix

Suppose \( T \in \mathcal{L}(V) \) and \( v_1, \ldots, v_n \) is a basis of \( V \). Then the following are equivalent:

1. The matrix of \( T \) with respect to \( v_1, \ldots, v_n \) is upper triangular;
2. \( T v_j \in \text{span}(v_1, \ldots, v_j) \) for each \( j = 1, \ldots, n \);
3. \( \text{span}(v_1, \ldots, v_j) \) is invariant under \( T \) for each \( j = 1, \ldots, n \).
**Upper-Triangular Matrices**

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**Definition: upper-triangular matrix**

A matrix is called *upper triangular* if all the entries below the diagonal equal 0.

*Example:*

$$ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix} $$

**Conditions for upper-triangular matrix**

Suppose $T \in L(V)$ and $v_1, \ldots, v_n$ is a basis of $V$. Then the following are equivalent:

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- $\text{span}(v_1, \ldots, v_j)$ is invariant under $T$ for each $j = 1, \ldots, n$. 

Over $\mathbb{C}$, every operator has an upper-triangular matrix

Suppose $V$ is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then $T$ has an upper-triangular matrix with respect to some basis of $V$.

\[
\mathcal{M}(T) = \begin{pmatrix}
\lambda_1 & * \\
& \lambda_2 \\
& & \ddots \\
& & & \lambda_n
\end{pmatrix}
\]

The first basis vector $v_1$ must be an eigenvector of $T$ with eigenvalue $\lambda_1$. 
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Over $\mathbb{C}$, Every Operator has Upper-Triangular Matrix
Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of $V$. Then the eigenvalues of $T$ are precisely the entries on the diagonal of that upper-triangular matrix.

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\mathcal{M}(T) = \begin{pmatrix}
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& \ldots \\
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Example: Define $T \in \mathcal{L}(\mathbb{R}^3)$ by $T(x, y, z) = (2x + y, 5y + 3z, 8z)$. Then $M(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$ with respect to the standard basis of $\mathbb{R}^3$. Thus the eigenvalues of $T$ are $2, 5,$ and $8$. 
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Determination of eigenvalues from upper-triangular matrix

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