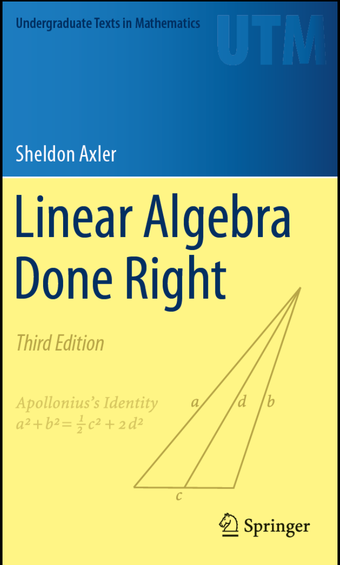
Subspaces



Undergraduate Texts in Mathematics
UTM
Sheldon Axler
Linear Algebra Done Right
Third Edition
Apollonius's Identity
$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$
a
d
b
c
Springer

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Example: $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbf{F}\}$ is a subspace of $\mathbf{F}^3$.

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- $0 \in U$; — additive identity
- $u, w \in U$ implies $u + w \in U$; — closed under addition
- $\lambda \in \mathbf{F}$ and $u \in U$ implies $\lambda u \in U$. — closed under scalar multiplication

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- The subspaces of $\mathbf{R}^3$ are precisely $\{0\}$, $\mathbf{R}^3$, all lines in $\mathbf{R}^3$ through the origin, and all planes in $\mathbf{R}^3$ through the origin.

Sums of Subspaces

Definition: *sum of subsets*

Suppose $U_1, \ldots, U_m$ are subsets of V. The *sum* of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}.$$

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Analogy: sums of subspaces $\longleftrightarrow$ unions of subsets in set theory

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- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

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Example: Suppose

$$U = \{(x, y, 0) \in \mathbf{F}^3 : x, y \in \mathbf{F}\} \quad \text{and} \quad W = \{(0, 0, z) \in \mathbf{F}^3 : z \in \mathbf{F}\}.$$

Then $\mathbf{F}^3 = U \oplus W$.

Condition for a direct sum

Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

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Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

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