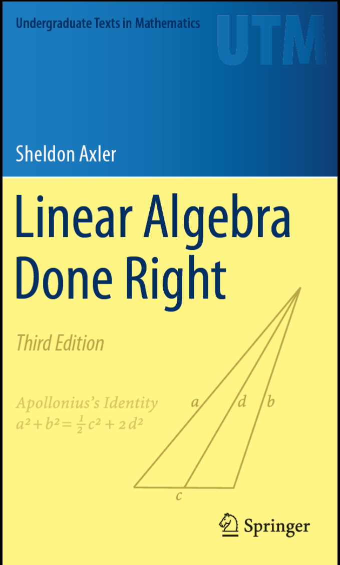
Square Roots of Operators



Example of Operator with No Square Root

Definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

Example of Operator with No Square Root

Definition: *square root*

An operator R is called a *square root* of an operator T if $R^2 = T$.

Example: Define $T \in \mathcal{L}(\mathbf{C}^3)$ by

$$T(z_1, z_2, z_3) = (z_2, z_3, 0).$$

Then T does not have a square root.

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m.

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Now

$$(I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1})^2$$

$$= I + 2a_1 N + (2a_2 + {a_1}^2) N^2 + (2a_3 + 2a_1 a_2) N^3 + \cdots$$

$$+ (2a_{m-1} + \text{terms involving } a_1, \ldots, a_{m-2}) N^{m-1}$$

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Now

$$(I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1})^2$$
$$= I + 2a_1 N + (2a_2 + {a_1}^2) N^2 + (2a_3 + 2a_1 a_2) N^3 + \cdots$$
$$+ (2a_{m-1} + \text{terms involving } a_1, \ldots, a_{m-2}) N^{m-1}$$
$$= I + N$$

if $2a_1 = 1$ (thus $a_1 = 1/2$)

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Now

$$\begin{aligned}(I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1})^2 \\ = I + 2a_1 N + (2a_2 + {a_1}^2) N^2 + (2a_3 + 2a_1 a_2) N^3 + \cdots \\ + (2a_{m-1} + \text{terms involving } a_1, \ldots, a_{m-2}) N^{m-1} \\ = I + N\end{aligned}$$

if $2a_1 = 1$ (thus $a_1 = 1/2$) and $2a_2 + {a_1}^2 = 0$ (thus $a_2 = -1/8$)

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Now

$$\begin{aligned}
&(I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1})^2 \\
&= I + 2a_1 N + (2a_2 + {a_1}^2) N^2 + (2a_3 + 2a_1 a_2) N^3 + \cdots \\
&\quad + (2a_{m-1} + \text{terms involving } a_1, \ldots, a_{m-2}) N^{m-1} \\
&= I + N
\end{aligned}$$

if $2a_1 = 1$ (thus $a_1 = 1/2$) and $2a_2 + {a_1}^2 = 0$ (thus $a_2 = -1/8$) and $a_3 = 1/16$

Identity Plus Nilpotent Has a Square Root

Identity plus nilpotent has a square root

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then $I + N$ has a square root.

Proof Consider the Taylor series for the function $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Because N is nilpotent, $N^m = 0$ for some positive integer m. Thus we guess that there is a square root of $I + N$ of the form

$$I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1}.$$

Now

$$\begin{aligned}(I + a_1 N + a_2 N^2 + a_3 N^3 + \cdots + a_{m-1} N^{m-1})^2 \\ = I + 2a_1 N + (2a_2 + {a_1}^2) N^2 + (2a_3 + 2a_1 a_2) N^3 + \cdots \\ + (2a_{m-1} + \text{terms involving } a_1, \ldots, a_{m-2}) N^{m-1} \\ = I + N\end{aligned}$$

if $2a_1 = 1$ (thus $a_1 = 1/2$) and $2a_2 + {a_1}^2 = 0$ (thus $a_2 = -1/8$) and $a_3 = 1/16$ and . . . . Continue in this fashion for $j = 4, \ldots, m-1$, at each step solving for a_j so that the coefficient of N^j on the right side of the equation above equals 0. ∎

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}\big(G(\lambda_j, T)\big)$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}\big(G(\lambda_j, T)\big)$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j \left(I + \frac{N_j}{\lambda_j} \right).$$

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}\big(G(\lambda_j, T)\big)$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j \Big(I + \frac{N_j}{\lambda_j}\Big).$$

Clearly N_j/λ_j is nilpotent, and so $I + N_j/\lambda_j$ has a square root.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}\big(G(\lambda_j, T)\big)$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j \Big(I + \frac{N_j}{\lambda_j}\Big).$$

Clearly N_j/λ_j is nilpotent, and so $I + N_j/\lambda_j$ has a square root.
Multiplying a square root of the number λ_j by a square root of $I + N_j/\lambda_j$ gives a square root R_j of $T|_{G(\lambda_j,T)}$.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}\big(G(\lambda_j, T)\big)$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j\Big(I + \frac{N_j}{\lambda_j}\Big).$$

Clearly N_j/λ_j is nilpotent, and so $I + N_j/\lambda_j$ has a square root. Multiplying a square root of the number λ_j by a square root of $I + N_j/\lambda_j$ gives a square root R_j of $T|_{G(\lambda_j,T)}$.

A typical vector $v \in V$ can be written uniquely in the form

$$v = u_1 + \cdots + u_m,$$

where each u_j is in $G(\lambda_j, T)$.

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}(G(\lambda_j, T))$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j\Big(I + \frac{N_j}{\lambda_j}\Big).$$

Clearly N_j/λ_j is nilpotent, and so $I + N_j/\lambda_j$ has a square root. Multiplying a square root of the number λ_j by a square root of $I + N_j/\lambda_j$ gives a square root R_j of $T|_{G(\lambda_j,T)}$.

A typical vector $v \in V$ can be written uniquely in the form

$$v = u_1 + \cdots + u_m,$$

where each u_j is in $G(\lambda_j, T)$.

Using this decomposition, define an operator $R \in \mathcal{L}(V)$ by

$$Rv = R_1 u_1 + \cdots + R_m u_m.$$

Over $\mathbf{C}$, Invertible Operators Have Square Roots

Over $\mathbf{C}$, invertible operators have square roots

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. For each j, there exists a nilpotent operator $N_j \in \mathcal{L}(G(\lambda_j, T))$ such that $T|_{G(\lambda_j,T)} = \lambda_j I + N_j$. Because T is invertible, none of the λ_j's equals 0, so we can write

$$T|_{G(\lambda_j,T)} = \lambda_j \Big(I + \frac{N_j}{\lambda_j} \Big).$$

Clearly N_j/λ_j is nilpotent, and so $I + N_j/\lambda_j$ has a square root. Multiplying a square root of the number λ_j by a square root of $I + N_j/\lambda_j$ gives a square root R_j of $T|_{G(\lambda_j,T)}$.

A typical vector $v \in V$ can be written uniquely in the form

$$v = u_1 + \cdots + u_m,$$

where each u_j is in $G(\lambda_j, T)$. Using this decomposition, define an operator $R \in \mathcal{L}(V)$ by

$$Rv = R_1 u_1 + \cdots + R_m u_m.$$

This operator R is a square root of T, completing the proof. ■

Linear Algebra Done Right, by Sheldon Axler