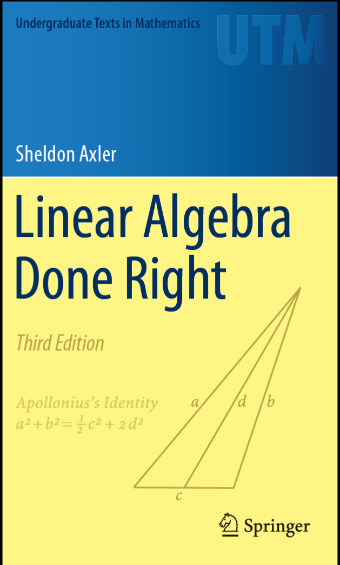
Self-Adjoint and Normal Operators, Part 2: Self-Adjoint Operators



Definition and Example of Self-Adjoint Operator

Definition: *self-adjoint*

An operator $T \in \mathcal{L}(V)$ is called *self-adjoint* if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

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Thus the operator T is self-adjoint if and only if $b = 3$.

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Thus $\lambda = \bar{\lambda}$, which means that λ is real, as desired. ∎

Over $\mathbf{C}$, $\langle Tv, v\rangle = 0$ for all v only for the 0 operator

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. If

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Conversely, if T is self-adjoint, then the right side of the last equation above equals 0, so $\langle Tv, v\rangle = \overline{\langle Tv, v\rangle}$ for every $v \in V$.

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If $T = T^*$ and $\langle Tv, v\rangle = 0$ for all v, then $T = 0$

Suppose T is a self-adjoint operator on V such that

$$\langle Tv, v\rangle = 0$$

for all $v \in V$. Then $T = 0$.

Linear Algebra Done Right, by Sheldon Axler