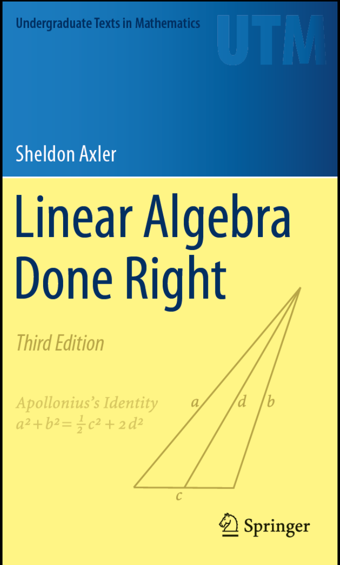
Products and Quotients of Vector Spaces, part 1: Products



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

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Product of vector spaces is a vector space

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$.

Example of Product of Vector Spaces

$$\mathbf{R}^2 \times \mathbf{R}^3 = \big\{\big((x_1, x_2), (x_3, x_4, x_5)\big) : x_1, x_2, x_3, x_4, x_5 \in \mathbf{R}\big\}$$

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The linear map $T\colon \mathbf{R}^2 \times \mathbf{R}^3 \to \mathbf{R}^5$ defined by

$$T((x_1, x_2), (x_3, x_4, x_5)) = (x_1, x_2, x_3, x_4, x_5)$$

is clearly an isomorphism of $\mathbf{R}^2 \times \mathbf{R}^3$ onto $\mathbf{R}^5$. Thus these two vector spaces are isomorphic.

Dimension of a Product

Dimension of a product is the sum of dimensions

Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Products and direct sums

Suppose that $U_1, \ldots, U_m$ are subspaces of V. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by

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A sum is a direct sum if and only if dimensions add up

Suppose V is finite-dimensional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m.$$

Linear Algebra Done Right, by Sheldon Axler