Preface for the Instructor

You are about to teach a course that will probably give students their second exposure to linear algebra. During their first brush with the subject, your students probably worked with Euclidean spaces and matrices. In contrast, this course will emphasize abstract vector spaces and linear maps.

The audacious title of this book deserves an explanation. Almost all linear algebra books use determinants to prove that every linear operator on a finite-dimensional complex vector space has an eigenvalue. Determinants are difficult, nonintuitive, and often defined without motivation. To prove the theorem about existence of eigenvalues on complex vector spaces, most books must define determinants, prove that a linear map is not invertible if and only if its determinant equals 0, and then define the characteristic polynomial. This tortuous (torturous?) path gives students little feeling for why eigenvalues exist.

In contrast, the simple determinant-free proofs presented here (for example, see 5.21) offer more insight. Once determinants have been banished to the end of the book, a new route opens to the main goal of linear algebra—understanding the structure of linear operators.

This book starts at the beginning of the subject, with no prerequisites other than the usual demand for suitable mathematical maturity. Even if your students have already seen some of the material in the first few chapters, they may be unaccustomed to working exercises of the type presented here, most of which require an understanding of proofs.

Here is a chapter-by-chapter summary of the highlights of the book:

- Chapter 1: Vector spaces are defined in this chapter, and their basic properties are developed.

- Chapter 2: Linear independence, span, basis, and dimension are defined in this chapter, which presents the basic theory of finite-dimensional vector spaces.

- Chapter 3: Linear maps are introduced in this chapter. The key result here is the Fundamental Theorem of Linear Maps (3.22): if T is a linear map on V, then $\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$. Quotient spaces and duality are topics in this chapter at a higher level of abstraction than other parts of the book; these topics can be skipped without running into problems elsewhere in the book.

- Chapter 4: The part of the theory of polynomials that will be needed to understand linear operators is presented in this chapter. This chapter contains no linear algebra. It can be covered quickly, especially if your students are already familiar with these results.

- Chapter 5: The idea of studying a linear operator by restricting it to small subspaces leads to eigenvectors in the early part of this chapter. The highlight of this chapter is a simple proof that on complex vector spaces, eigenvalues always exist. This result is then used to show that each linear operator on a complex vector space has an upper-triangular matrix with respect to some basis. All this is done without defining determinants or characteristic polynomials!

- Chapter 6: Inner product spaces are defined in this chapter, and their basic properties are developed along with standard tools such as orthonormal bases and the Gram–Schmidt Procedure. This chapter also shows how orthogonal projections can be used to solve certain minimization problems.

- Chapter 7: The Spectral Theorem, which characterizes the linear operators for which there exists an orthonormal basis consisting of eigenvectors, is the highlight of this chapter. The work in earlier chapters pays off here with especially simple proofs. This chapter also deals with positive operators, isometries, the Polar Decomposition, and the Singular Value Decomposition.

- Chapter 8: Minimal polynomials, characteristic polynomials, and generalized eigenvectors are introduced in this chapter. The main achievement of this chapter is the description of a linear operator on a complex vector space in terms of its generalized eigenvectors. This description enables one to prove many of the results usually proved using Jordan Form. For example, these tools are used to prove that every invertible linear operator on a complex vector space has a square root. The chapter concludes with a proof that every linear operator on a complex vector space can be put into Jordan Form.

- Chapter 9: Linear operators on real vector spaces occupy center stage in this chapter. Here the main technique is complexification, which is a natural extension of an operator on a real vector space to an operator on a complex vector space. Complexification allows our results about complex vector spaces to be transferred easily to real vector spaces. For example, this technique is used to show that every linear operator on a real vector space has an invariant subspace of dimension 1 or 2. As another example, we show that that every linear operator on an odd-dimensional real vector space has an eigenvalue.

- Chapter 10: The trace and determinant (on complex vector spaces) are defined in this chapter as the sum of the eigenvalues and the product of the eigenvalues, both counting multiplicity. These easy-to-remember definitions would not be possible with the traditional approach to eigenvalues, because the traditional method uses determinants to prove that sufficient eigenvalues exist. The standard theorems about determinants now become much clearer. The Polar Decomposition and the Real Spectral Theorem are used to derive the change of variables formula for multivariable integrals in a fashion that makes the appearance of the determinant there seem natural.

This book usually develops linear algebra simultaneously for real and complex vector spaces by letting $\mathbf{F}$ denote either the real or the complex numbers. If you and your students prefer to think of $\mathbf{F}$ as an arbitrary field, then see the comments at the end of Section 1.A. I prefer avoiding arbitrary fields at this level because they introduce extra abstraction without leading to any new linear algebra. Also, students are more comfortable thinking of polynomials as functions instead of the more formal objects needed for polynomials with coefficients in finite fields. Finally, even if the beginning part of the theory were developed with arbitrary fields, inner product spaces would push consideration back to just real and complex vector spaces.

You probably cannot cover everything in this book in one semester. Going through the first eight chapters is a good goal for a one-semester course. If you must reach Chapter 10, then consider covering Chapters 4 and 9 in fifteen minutes each, as well as skipping the material on quotient spaces and duality in Chapter 3.

A goal more important than teaching any particular theorem is to develop in students the ability to understand and manipulate the objects of linear algebra. Mathematics can be learned only by doing. Fortunately, linear algebra has many good homework exercises. When teaching this course, during each class I usually assign as homework several of the exercises, due the next class. Going over the homework might take up a third or even half of a typical class.

Major changes from the previous edition:

- This edition contains 561 exercises, including 337 new exercises that were not in the previous edition. Exercises now appear at the end of each section, rather than at the end of each chapter.
- Many new examples have been added to illustrate the key ideas of linear algebra.
- Beautiful new formatting, including the use of color, creates pages with an unusually pleasant appearance in both print and electronic versions. As a visual aid, definitions are in beige boxes and theorems are in blue boxes (in color versions of the book).
- Each theorem now has a descriptive name.
- New topics covered in the book include product spaces, quotient spaces, and duality.
- Chapter 9 (Operators on Real Vector Spaces) has been completely rewritten to take advantage of simplifications via complexification. This approach allows for more streamlined presentations in Chapters 5 and 7 because those chapters now focus mostly on complex vector spaces.
- Hundreds of improvements have been made throughout the book. For example, the proof of Jordan Form (Section 8.D) has been simplified.

Please check the website below for additional information about the book. I may occasionally write new sections on additional topics. These new sections will be posted on the website. Your suggestions, comments, and corrections are most welcome.

Best wishes for teaching a successful linear algebra class!

Sheldon Axler
Mathematics Department
San Francisco State University
San Francisco, CA 94132, USA

website: `linear.axler.net`
e-mail: `linear@axler.net`
Twitter: `@AxlerLinear`