\( F \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \).

\( V \) denotes a finite-dimensional inner product space over \( F \).
Definition: *positive operator*

An operator $T \in \mathcal{L}(V)$ is called *positive* if $T$ is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$. 

If $V$ is a complex vector space, then the requirement that $T$ is self-adjoint can be dropped from the definition above.

Examples:

- If $U$ is a subspace of $V$, then the orthogonal projection $P_U$ is a positive operator.
- If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 \leq 4c$, then $T^2 + bT + cI$ is a positive operator.
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An operator $R$ is called a *square root* of an operator $T$ if $R^2 = T$. 

Example: If $T \in L(F^3)$ is defined by $T(z_1, z_2, z_3) = (z_3, 0, 0)$, then the operator $R \in L(F^3)$ defined by $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of $T$. 

Example: If $T \in L(F^3)$ has matrix $M(T) = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, then the operator $R \in L(F^3)$ with matrix $M(R) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, is a square root of $T$. 

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We will prove that (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (e) $\implies$ (a).

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Let $v$ be an eigenvector of $T$ corresponding to $\lambda$. Then

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Let $R \in \mathcal{L}(V)$ be such that $Re_j = \sqrt{\lambda_j}e_j$ for $j = 1, \ldots, n$. Then $R$ is a positive operator. Furthermore, $R^2 e_j = \lambda_j e_j = Te_j$ for each $j$, which implies that $R^2 = T$. Thus $R$ is a positive square root of $T$. Hence (c) holds.
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Finally, suppose (e) holds. Let $R \in \mathcal{L}(V)$ be such that $T = R^*R$. Then

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Hence $T$ is self-adjoint.

To complete the proof that (a) holds, note that

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(e) there exists $R \in \mathcal{L}(V)$ such that $T = R^*R$.

Finally, suppose (e) holds. Let $R \in \mathcal{L}(V)$ be such that $T = R^*R$. Now

\[ T^* = (R^*R)^* = R^*(R^*)^* = R^*R = T. \]

Hence $T$ is self-adjoint. To complete the proof that (a) holds, note that

\[ \langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle \geq 0 \]

for every $v \in V$. 
Characterization of positive operators

Let \( T \in \mathcal{L}(V) \). Then the following are equivalent:

(a) \( T \) is positive;
(b) \( T \) is self-adjoint and all eigenvalues of \( T \) are nonnegative;
(c) \( T \) has a positive square root;
(d) \( T \) has a self-adjoint square root;
(e) there exists \( R \in \mathcal{L}(V) \) such that \( T = R^*R \).

Finally, suppose (e) holds. Let \( R \in \mathcal{L}(V) \) be such that \( T = R^*R \). Now

\[
T^* = (R^*R)^* \\
= R^*(R^*)^* \\
= R^*R \\
= T.
\]

Hence \( T \) is self-adjoint. To complete the proof that (a) holds, note that

\[
\langle Tv, v \rangle = \langle R^*Rv, v \rangle \\
= \langle Rv, Rv \rangle \\
\geq 0
\]

for every \( v \in V \). Thus \( T \) is positive. \( \blacksquare \)
Uniqueness of Positive Square Root

Each positive operator has only one positive square root

Every positive operator on $V$ has a unique positive square root.
Each positive operator has only one positive square root

Every positive operator on $V$ has a unique positive square root.

A positive operator can have infinitely many square roots, although only one of them can be positive. For example, the identity operator on $V$ has infinitely many square roots if $\dim V > 1$. 