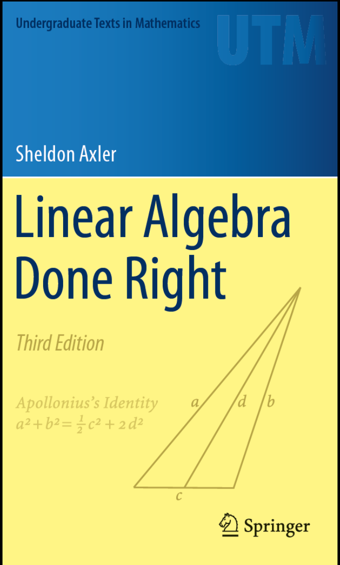
Orthonormal Bases



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes an inner product space over $\mathbf{F}$.

Orthonormal Lists

Definition: *orthonormal*

- A list of vectors in V is called *orthonormal* if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.
- In other words, a list $e_1, \ldots, e_m$ of vectors in V is orthonormal if

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- The standard basis in $\mathbf{F}^n$ is an orthonormal list.
- $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is an orthonormal list in $\mathbf{F}^3$.

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- $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$ is an orthonormal list in $\mathbf{F}^3$.

The norm of an orthonormal linear combination

If $e_1, \ldots, e_m$ is an orthonormal list of vectors in V, then

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \ldots, a_m \in \mathbf{F}$.

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Proof Suppose $e_1, \ldots, e_m$ is an orthonormal list of vectors in V and $a_1, \ldots, a_m \in \mathbf{F}$ are such that

$$a_1 e_1 + \cdots + a_m e_m = 0.$$

Then $|a_1|^2 + \cdots + |a_m|^2 = 0$, which means that all the a_j's are 0. Thus $e_1, \ldots, e_m$ is linearly independent. ∎

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Example:

$$\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right),\left(\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right),\left(\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right),\left(-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}\right)$$

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is an orthonormal basis of $\mathbf{F}^4$.

Writing a vector as linear combination of orthonormal basis

Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v \in V$. Then

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

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Proof There exist scalars $a_1,\ldots,a_n$ such that $v = a_1e_1 + \cdots + a_ne_n$. Take inner product with e_j.

Gram–Schmidt Procedure

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Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $e_1 = v_1/\|v_1\|$. For $j = 2, \ldots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}.$$

Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_j) = \operatorname{span}(e_1, \ldots, e_j)$$

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Example:

Suppose $V = \mathcal{P}_2(\mathbf{R})$, where the inner product is given by

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Applying the Gram–Schmidt Procedure to the basis $1, x, x^2$ produces the orthonormal basis

$$\sqrt{\tfrac{1}{2}},\ \sqrt{\tfrac{3}{2}}x,\ \sqrt{\tfrac{45}{8}}\big(x^2 - \tfrac{1}{3}\big).$$

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Orthonormal list extends to orthonormal basis

Suppose V is finite-dimensional. Then every orthonormal list of vectors in V can be extended to an orthonormal basis of V.

Schur's Theorem

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some orthonormal basis of V.

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Proof T has an upper-triangular matrix with respect to some basis $v_1, \ldots, v_n$ of V:

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Apply the Gram–Schmidt Procedure to $v_1, \ldots, v_n$, producing an orthonormal basis $e_1, \ldots, e_n$ of V.

Because

$$\operatorname{span}(e_1, \ldots, e_j) = \operatorname{span}(v_1, \ldots, v_j)$$

for each j, we conclude that $\operatorname{span}(e_1, \ldots, e_j)$ is invariant under T for each $j = 1, \ldots, n$. Thus T has an upper-triangular matrix with respect to the orthonormal basis $e_1, \ldots, e_n$. ∎

Linear Functionals

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Proof Let $e_1, \dots, e_n$ be an orthonormal basis of V. Then

$$u = \overline{\varphi(e_1)}e_1 + \cdots + \overline{\varphi(e_n)}e_n.$$

Linear Algebra Done Right, by Sheldon Axler