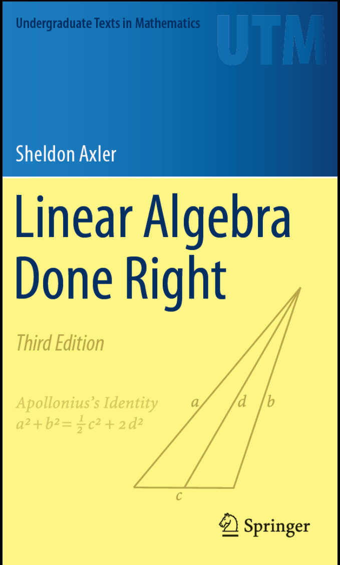
Normal Operators on Real Inner Product Spaces



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero inner product space over $\mathbf{F}$.

Normal but not Self-Adjoint Operators

Normal but not self-adjoint operators

Suppose $\dim V = 2$, $\mathbf{F} = \mathbf{R}$, and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal but not self-adjoint.

(b) The matrix of T with respect to every orthonormal basis of V has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

with $b \neq 0$.

(c) The matrix of T with respect to some orthonormal basis of V has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

with $b > 0$.

Normal Operators and Invariant Subspaces

Normal operators and invariant subspaces

Suppose $T \in \mathcal{L}(V)$ is normal and U is a subspace of V that is invariant under T. Then

(a) $U^\perp$ is invariant under T;

(b) U is invariant under T^*;

(c) $(T|_U)^* = (T^*)|_U$;

(d) $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal operators.

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Proof Let $e_1, \ldots, e_m$ be an orthonormal basis of U.

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Proof Let $e_1, \ldots, e_m$ be an orthonormal basis of U. Extend to an orthonormal basis $e_1, \ldots, e_m, f_1, \ldots, f_n$ of V.

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$$\mathcal{M}(T) = \begin{array}{c} \\ e_1 \\ \vdots \\ e_m \\ f_1 \\ \vdots \\ f_n \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & \ldots & e_m & f_1 & \ldots & f_n \end{array} \\ \left(\begin{array}{cc} A & B \\ & \\ 0 & C \end{array} \right) \end{array}.$$

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Because T is normal,

$$\sum_{j=1}^{m} \|Te_j\|^2 = \sum_{j=1}^{m} \|T^* e_j\|^2.$$

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Thus B is the matrix of all 0's, proving (a).

Description of Normal Operators on Real Inner Product Spaces

Characterization of normal operators when $\mathbf{F} = \mathbf{R}$

Suppose V is a real inner product space and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) There is an orthonormal basis of V with respect to which T has a block diagonal matrix such that each block is a 1-by-1 matrix or a 2-by-2 matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

with $b > 0$.

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