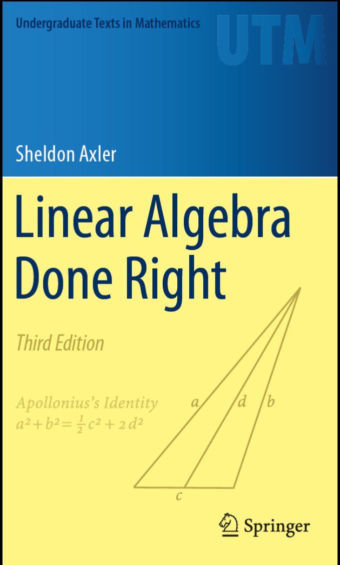
The Minimal Polynomial



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Example: The polynomial

$$2 + 9z^2 + z^7$$

is a monic polynomial of degree 7.

Existence of Minimal Polynomial

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is not linearly independent in $\mathcal{L}(V)$, because $\mathcal{L}(V)$ has dimension n^2 and the list has length $n^2 + 1$.

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The Linear Dependence Lemma implies that T^m is a linear combination of $I, T, T^2, \ldots, T^{m-1}$. Thus there exist scalars $a_0, a_1, a_2, \ldots, a_{m-1} \in \mathbf{F}$ such that $a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1} T^{m-1} + T^m = 0$.

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$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{m-1} T^{m-1} + T^m = 0.$$

Define a monic polynomial $p \in \mathcal{P}(\mathbf{F})$ by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{m-1} z^{m-1} + z^m.$$

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Then $p(T) = 0$.

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Then $p(T) = 0$.

No monic polynomial $q \in \mathcal{P}(\mathbf{F})$ with degree smaller than m can satisfy $q(T) = 0$. Suppose $q \in \mathcal{P}(\mathbf{F})$ is a monic polynomial with degree m and $q(T) = 0$. Then $(p - q)(T) = 0$ and $\deg(p - q) < m$. Thus $q = p$, completing the proof. ∎

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Example: Let T be the operator on $\mathbf{C}^5$ whose matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -3 \\ 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

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We have

$$3\mathcal{M}(I) - 6\mathcal{M}(T) = -\mathcal{M}(T)^5$$

with no solutions for lower powers. Thus the minimal polynomial of T is

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For general $T \in \mathcal{L}(V)$, consider the system of linear equations

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for successive values of $m = 1, 2, \ldots$ until this system of equations has a solution $a_0, a_1, a_2, \ldots, a_{m-1}$.

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for successive values of $m = 1, 2, \ldots$ until this system of equations has a solution $a_0, a_1, a_2, \ldots, a_{m-1}$. The scalars $a_0, a_1, a_2, \ldots, a_{m-1}, 1$ will then be the coefficients of the minimal polynomial of T.

Polynomials That Annihilate T

$q(T) = 0$ implies q is a multiple of the minimal polynomial

Suppose $T \in \mathcal{L}(V)$ and $q \in \mathcal{P}(\mathbf{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.

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To prove the other direction, now suppose $q(T) = 0$. By the Division Algorithm for Polynomials, there exist polynomials $s, r \in \mathcal{P}(\mathbf{F})$ such that

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Characteristic polynomial is a multiple of minimal polynomial

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

Zeros of the Minimal Polynomial

Eigenvalues are the zeros of the minimal polynomial

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Now for every $v \in V$ we have

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We have shown that every zero of p is an eigenvalue of T.

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for every nonnegative integer j. Thus

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Let $T \in \mathcal{L}(V)$. Then the zeros of the minimal polynomial of T are precisely the eigenvalues of T.

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