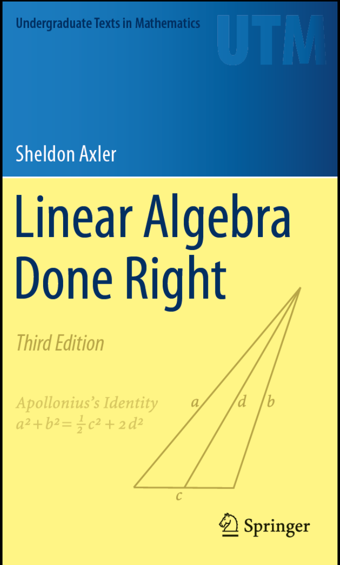
Matrices, part 2: Matrix Multiplication



Motivation for Definition of Matrix Multiplication

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Suppose $v_1, \ldots, v_n$ is a basis of V
and $w_1, \ldots, w_m$ is a basis of W.
Suppose $u_1, \ldots, u_p$ is a basis of U.

Consider linear maps $T\colon U \to V$
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Does $\mathcal{M}(ST)$ equal $\mathcal{M}(S)\mathcal{M}(T)$?

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For $1 \leq k \leq p$, we have

$$\begin{aligned}(ST)u_k &= S\Big(\sum_{r=1}^{n} C_{r,k} v_r\Big) \\ &= \sum_{r=1}^{n} C_{r,k} S v_r \\ &= \sum_{r=1}^{n} C_{r,k} \sum_{j=1}^{m} A_{j,r} w_j \\ &= \sum_{j=1}^{m} \Big(\sum_{r=1}^{n} A_{j,r} C_{r,k}\Big) w_j.\end{aligned}$$

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Thus $\mathcal{M}(ST)$ is the m-by-p matrix whose entry in row j, column k, is

$$\sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

Definition of Matrix Multiplication

Definition: *matrix multiplication*

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

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Example: $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 10 & 7 & 4 & 1 \\ 26 & 19 & 12 & 5 \\ 42 & 31 & 20 & 9 \end{pmatrix}$$

Matrices and Products of Linear Maps

In the following result, the same basis of V is used in considering $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, the same basis of W is used in considering $S \in \mathcal{L}(V, W)$ and $ST \in \mathcal{L}(U, W)$, and the same basis of U is used in considering $T \in \mathcal{L}(U, V)$ and $ST \in \mathcal{L}(U, W)$.

The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

The proof of the result above is the calculation that was done as motivation before the definition of matrix multiplication.

Notation for Row and Column of Matrix

Notation: $A_{j,\cdot}$, $A_{\cdot,k}$

Suppose A is an m-by-n matrix.

- If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of A.
- If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of A.

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Example: If $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$, then $A_{2,\cdot}$ is row 2 of A and $A_{\cdot,2}$ is column 2 of A. In other words,

$$A_{2,\cdot} = \begin{pmatrix} 1 & 9 & 7 \end{pmatrix} \quad \text{and} \quad A_{\cdot,2} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Alternative Ways to Think about Matrix Multiplication

Example: $\begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 26 \end{pmatrix} = 26.$

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Entry of matrix product equals row times column

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot}\, C_{\cdot,k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$.

Column of matrix product equals matrix times column

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$.

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Column of matrix product equals matrix times column

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$.

Row of matrix product equals row times matrix

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then

$$(AC)_{j,\cdot} = A_{j,\cdot}C$$

for $1 \leq k \leq p$.

Matrix Times a Column Vector

Linear combination of columns

Suppose A is an m-by-n matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n-by-1 matrix.

Then $Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.

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Example:

$$\begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 3 \begin{pmatrix} 8 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 4 \\ 9 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

$$= \begin{pmatrix} 30 \\ 8 \end{pmatrix}$$

Row Vector Times a Matrix

Linear combination of rows

Suppose $a = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ is a 1-by-n matrix and C is an n-by-p matrix. Then

$$aC = a_1 C_{1,\cdot} + \cdots + a_n C_{n,\cdot} .$$

In other words, aC is a linear combination of the rows of C, with the scalars that multiply the rows coming from a.

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Example:

$$\begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix} = 3 \begin{pmatrix} 8 & 4 & 5 \end{pmatrix} + (-1) \begin{pmatrix} 1 & 9 & 7 \end{pmatrix}$$
$$= \begin{pmatrix} 23 & 3 & 8 \end{pmatrix}$$

Linear Algebra Done Right, by Sheldon Axler