Matrices, part 1: The Matrix of a Linear Map
Notation

\( \mathbb{F} \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \).

\( V \) and \( W \) denote vector spaces over \( \mathbb{F} \).
**Definition: matrix, \( A_{j,k} \)**

Let \( m \) and \( n \) denote positive integers. An \( m \)-by-\( n \) matrix \( A \) is a rectangular array of elements of \( F \) with \( m \) rows and \( n \) columns:

\[
A = \begin{pmatrix}
A_{1,1} & \ldots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{m,1} & \ldots & A_{m,n}
\end{pmatrix}.
\]

The notation \( A_{j,k} \) denotes the entry in row \( j \), column \( k \) of \( A \).
Definition: *matrix, $A_{j,k}$*

Let $m$ and $n$ denote positive integers. An $m$-by-$n$ *matrix* $A$ is a rectangular array of elements of $F$ with $m$ rows and $n$ columns:

$$
A = \begin{pmatrix}
A_{1,1} & \cdots & A_{1,n} \\
\vdots & & \vdots \\
A_{m,1} & \cdots & A_{m,n}
\end{pmatrix}.
$$

The notation $A_{j,k}$ denotes the entry in row $j$, column $k$ of $A$.

The first index refers to the row number and the second index refers to the column number.
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Thus \( A_{2,3} \) refers to the entry in the second row, third column of \( A \).
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**Example:** Suppose \(A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}\). Then \(A_{2,3} = 7\).
**Definition: matrix of a linear map, \( \mathcal{M}(T) \)**

Suppose \( T \in \mathcal{L}(V, W) \) and \( v_1, \ldots, v_n \) is a basis of \( V \) and \( w_1, \ldots, w_m \) is a basis of \( W \). The *matrix of \( T \) with respect to these bases* is the \( m \)-by-\( n \) matrix \( \mathcal{M}(T) \) whose entries \( A_{j,k} \) are defined by

\[
Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.
\]

If the bases are not clear from the context, then the notation \( \mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)) \) is used.
Understanding the Matrix of a Linear Map

\[ \mathcal{M}(T) = \begin{pmatrix} w_1 & \vdots & A_{1,k} & \vdots & \cdots & \cdots & \cdots & A_{m,k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_m & \vdots & A_{m,k} \end{pmatrix} \]

The \( k \)th column of \( \mathcal{M}(T) \) consists of the scalars needed to write \( Tv_k \) as a linear combination of \( w_1, \ldots, w_m \):

\[ Tv_k = \sum_{j=1}^{m} A_{j,k} w_j. \]

The picture above should remind you that \( Tv_k \) can be computed from \( \mathcal{M}(T) \) by multiplying each entry in the \( k \)th column by the corresponding \( w_j \) from the left column, and then adding up the resulting vectors.
Understanding the Matrix of a Linear Map

\[ M(T) = \begin{pmatrix}
    v_1 & \ldots & v_k & \ldots & v_n \\
    w_1 & \vdots & A_{1,k} & \vdots & \vdots \\
    w_m & \vdots & A_{m,k} & \vdots & \vdots
\end{pmatrix}. \]

The \( k \)th column of \( M(T) \) consists of the scalars needed to write \( Tv_k \) as a linear combination of \( w_1, \ldots, w_m \):

\[ Tv_k = \sum_{j=1}^{m} A_{j,k}w_j. \]
The $k^{th}$ column of $\mathcal{M}(T)$ consists of the scalars needed to write $Tv_k$ as a linear combination of $w_1, \ldots, w_m$:

$$Tv_k = \sum_{j=1}^{m} A_{j,k}w_j.$$
Example: Suppose $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ is defined by 

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

Because $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$, the matrix of $T$ with respect to the standard bases is the 3-by-2 matrix 

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$
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$$M(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

Example: Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Because $(x^n)' = nx^{n-1}$, the matrix of $D$ with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ is the 3-by-4 matrix

$$M(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$
Addition of Matrices

**Definition: matrix addition**

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

\[
\begin{pmatrix}
A_{1,1} & \ldots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{m,1} & \ldots & A_{m,n}
\end{pmatrix}
+ 
\begin{pmatrix}
C_{1,1} & \ldots & C_{1,n} \\
\vdots & \ddots & \vdots \\
C_{m,1} & \ldots & C_{m,n}
\end{pmatrix}
= 
\begin{pmatrix}
A_{1,1} + C_{1,1} & \ldots & A_{1,n} + C_{1,n} \\
\vdots & \ddots & \vdots \\
A_{m,1} + C_{m,1} & \ldots & A_{m,n} + C_{m,n}
\end{pmatrix}.
\]

In other words, \((A + C)_{j,k} = A_{j,k} + C_{j,k}\).
Addition of Matrices

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\end{pmatrix}
+ 
\begin{pmatrix}
C_{1,1} & \ldots & C_{1,n} \\
\vdots & & \vdots \\
C_{m,1} & \ldots & C_{m,n}
\end{pmatrix}
= 
\begin{pmatrix}
A_{1,1} + C_{1,1} & \ldots & A_{1,n} + C_{1,n} \\
\vdots & & \vdots \\
A_{m,1} + C_{m,1} & \ldots & A_{m,n} + C_{m,n}
\end{pmatrix}.
\]

In other words, \((A + C)_{j,k} = A_{j,k} + C_{j,k}\).

In the following result, the assumption is that the same bases are used for \(\mathcal{M}(S + T)\), \(\mathcal{M}(S)\), and \(\mathcal{M}(T)\).
Addition of Matrices

**Definition:** *matrix addition*

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

\[
\begin{pmatrix}
A_{1,1} & \ldots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{m,1} & \ldots & A_{m,n}
\end{pmatrix} +
\begin{pmatrix}
C_{1,1} & \ldots & C_{1,n} \\
\vdots & \ddots & \vdots \\
C_{m,1} & \ldots & C_{m,n}
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\begin{pmatrix}
A_{1,1} + C_{1,1} & \ldots & A_{1,n} + C_{1,n} \\
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\end{pmatrix}.
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In other words, \((A + C)_{j,k} = A_{j,k} + C_{j,k}\).

In the following result, the assumption is that the same bases are used for \(\mathcal{M}(S + T)\), \(\mathcal{M}(S)\), and \(\mathcal{M}(T)\).

**The matrix of the sum of linear maps**

Suppose \(S, T \in \mathcal{L}(V, W)\). Then \(\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)\).
Definition: *scalar multiplication of a matrix*

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

\[
\lambda \left( \begin{array}{ccc}
A_{1,1} & \cdots & A_{1,n} \\
\vdots & \ddots & \vdots \\
A_{m,1} & \cdots & A_{m,n}
\end{array} \right) = \left( \begin{array}{ccc}
\lambda A_{1,1} & \cdots & \lambda A_{1,n} \\
\vdots & \ddots & \vdots \\
\lambda A_{m,1} & \cdots & \lambda A_{m,n}
\end{array} \right).
\]

In other words, \((\lambda A)_{j,k} = \lambda A_{j,k}\).
**Definition: **scalar multiplication of a matrix

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \ldots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \ldots & \lambda A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda A_{m,1} & \ldots & \lambda A_{m,n} \end{pmatrix}.$$  

In other words, $$(\lambda A)_{j,k} = \lambda A_{j,k}.$$  

In the following result, the assumption is that the same bases are used for $\mathcal{M}(\lambda T)$ and $\mathcal{M}(T).$
Definition: **scalar multiplication of a matrix**

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

\[
\lambda \begin{pmatrix}
A_{1,1} & \ldots & A_{1,n} \\
\vdots & & \vdots \\
A_{m,1} & \ldots & A_{m,n}
\end{pmatrix} = \begin{pmatrix}
\lambda A_{1,1} & \ldots & \lambda A_{1,n} \\
\vdots & & \vdots \\
\lambda A_{m,1} & \ldots & \lambda A_{m,n}
\end{pmatrix}.
\]

In other words, \((\lambda A)_{j,k} = \lambda A_{j,k}\).

In the following result, the assumption is that the same bases are used for \(\mathcal{M}(\lambda T)\) and \(\mathcal{M}(T)\).

**The matrix of a scalar times a linear map**

Suppose \(\lambda \in \mathbb{F}\) and \(T \in \mathcal{L}(V, W)\). Then \(\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)\).
### The Vector Space of Matrices

**Notation:** $\mathbb{F}^{m,n}$

For $m$ and $n$ positive integers, the set of all $m$-by-$n$ matrices with entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m,n}$. 

$\dim \mathbb{F}^{m,n} = mn$
**Notation:** $F^{m,n}$

For $m$ and $n$ positive integers, the set of all $m$-by-$n$ matrices with entries in $F$ is denoted by $F^{m,n}$.

\[
\dim F^{m,n} = mn
\]

Suppose $m$ and $n$ are positive integers. With addition and scalar multiplication defined as above, $F^{m,n}$ is a vector space with dimension $mn$. 