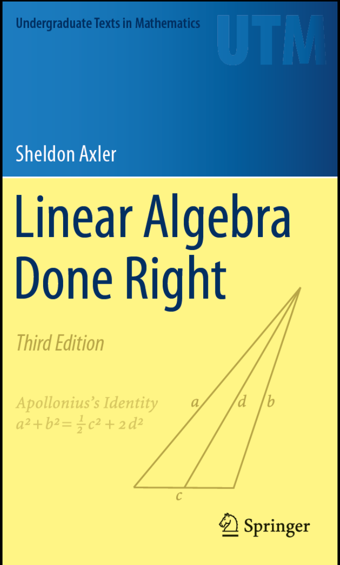
The Vector Space of Linear Maps



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

Notation: $\mathcal{P}(\mathbf{F})$

$\mathcal{P}(\mathbf{F})$ is the vector space of all polynomials with coefficients in $\mathbf{F}$.

Definition of Linear Map

Definition: *linear map*

A *linear map* from V to W is a function $T\colon V \to W$ with the following properties:

Definition of Linear Map

Definition: *linear map*

A *linear map* from V to W is a function $T: V \to W$ with the following properties:

- additivity: $T(u+v) = Tu + Tv$ for all $u, v \in V$;

Definition of Linear Map

Definition: *linear map*

A *linear map* from V to W is a function $T\colon V \to W$ with the following properties:

- additivity: $T(u+v) = Tu + Tv$ for all $u, v \in V$;
- homogeneity: $T(\lambda u) = \lambda(Tu)$ for all $\lambda \in \mathbf{F}$ and all $u \in V$.

Definition of Linear Map

Definition: *linear map*

A *linear map* from V to W is a function $T: V \to W$ with the following properties:

- additivity: $T(u_1 + u_2) = Tu_1 + Tu_2$ for all $u_1, u_2 \in V$;
- homogeneity: $T(\lambda u) = \lambda(Tu)$ for all $\lambda \in \mathbf{F}$ and all $u \in V$.

For linear maps, we often use the notation Tu as well as the more standard functional notation $T(u)$.

Definition of Linear Map

Definition: *linear map*

A *linear map* from V to W is a function $T: V \to W$ with the following properties:

- additivity: $T(u_1 + u_2) = Tu_1 + Tu_2$ for all $u_1, u_2 \in V$;
- homogeneity: $T(\lambda u) = \lambda(Tu)$ for all $\lambda \in \mathbf{F}$ and all $u \in V$.

For linear maps, we often use the notation Tu as well as the more standard functional notation $T(u)$.

Notation: $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.
- Differentiation: Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by $Dp = p'$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.
- Differentiation: Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by $Dp = p'$.
- Integration: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathbf{R}\big)$ by $Tp = \int_0^1 p(x)\,dx$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.
- Differentiation: Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by $Dp = p'$.
- Integration: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathbf{R}\big)$ by $Tp = \displaystyle\int_0^1 p(x)\,dx$.
- Multiplication by x^2: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by

$$(Tp)(x) = x^2 p(x)$$

for $x \in \mathbf{R}$.

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.
- Differentiation: Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by $Dp = p'$.
- Integration: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathbf{R}\big)$ by $Tp = \int_0^1 p(x)\,dx$.
- Multiplication by x^2: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by
$$(Tp)(x) = x^2 p(x)$$
for $x \in \mathbf{R}$.
- Backward shift: Define $T \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ by
$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

Examples of Linear Maps

- Zero: Define $0 \in \mathcal{L}(V, W)$ by $0u = 0$ for all $u \in V$.
- Identity map: Define $I \in \mathcal{L}(V, V)$ by $Iu = u$ for all $u \in V$.
- Differentiation: Define $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by $Dp = p'$.
- Integration: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathbf{R}\big)$ by $Tp = \int_0^1 p(x)\,dx$.
- Multiplication by x^2: Define $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ by
$$(Tp)(x) = x^2 p(x)$$
for $x \in \mathbf{R}$.
- Backward shift: Define $T \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ by
$$T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$
- From $\mathbf{R}^3$ to $\mathbf{R}^2$: Define $T \in \mathcal{L}(\mathbf{R}^3, \mathbf{R}^2)$ by
$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

Linear Map Determined By What It Does On a Basis

Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Linear Map Determined By What It Does On a Basis

Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Proof

Define $T\colon V \to W$ by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n,$$

where $c_1, \ldots, c_n$ are arbitrary elements of $\mathbf{F}$. ∎

Algebraic Operations on $\mathcal{L}(V, W)$

Definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(u) = Su + Tu \quad \text{and} \quad (\lambda T)(u) = \lambda(Tu)$$

for all $u \in V$.

Algebraic Operations on $\mathcal{L}(V, W)$

Definition: *addition and scalar multiplication on $\mathcal{L}(V, W)$*

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(u) = Su + Tu \quad \text{and} \quad (\lambda T)(u) = \lambda(Tu)$$

for all $u \in V$.

$\mathcal{L}(V, W)$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Composition of Linear Maps

Definition: *Product of Linear Maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

Composition of Linear Maps

Definition: *Product of Linear Maps*

If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then the *product* $ST \in \mathcal{L}(U,W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

Algebraic properties of products of linear maps

- Associativity: $(T_1T_2)T_3 = T_1(T_2T_3)$

Composition of Linear Maps

Definition: ***Product of Linear Maps***

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

Algebraic properties of products of linear maps

- Associativity: $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity: $TI = IT = T$

Composition of Linear Maps

Definition: *Product of Linear Maps*

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$.

Algebraic properties of products of linear maps

- Associativity: $(T_1T_2)T_3 = T_1(T_2T_3)$
- Identity: $TI = IT = T$
- Distributive Properties:
  $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$

0 Goes to 0 Under a Linear Map

Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

0 Goes to 0 Under a Linear Map

Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof

By additivity, we have

$$T(0) = T(0 + 0) = T(0) + T(0).$$

0 Goes to 0 Under a Linear Map

Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof

By additivity, we have

$$T(0) = T(0 + 0) = T(0) + T(0).$$

Add the additive inverse of $T(0)$ to each side of the equation above to conclude that $0 = T(0)$. ∎

Linear Algebra Done Right, by Sheldon Axler