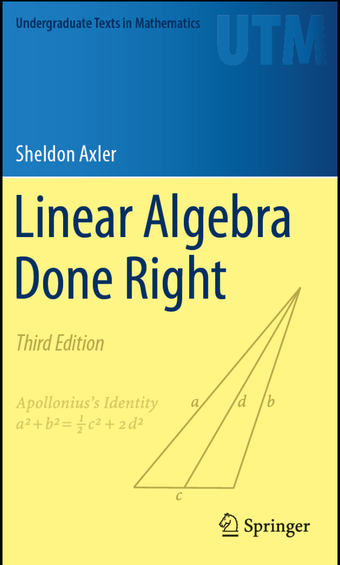
Jordan Form



Undergraduate Texts in Mathematics
UTM
Sheldon Axler
Linear Algebra Done Right
Third Edition
Apollonius's Identity
$a^2 + b^2 = \frac{1}{2}c^2 + 2d^2$
a
d
b
c
Springer

Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Examples

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Basis corresponding to a nilpotent operator

Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there exist vectors $v_1, \ldots, v_n \in V$ and nonnegative integers $m_1, \ldots, m_n$ such that

- $N^{m_1}v_1, \ldots, Nv_1, v_1, \ldots, N^{m_n}v_n, \ldots, Nv_n, v_n$ is a basis of V;
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Jordan Basis and Jordan Form

Definition: *Jordan basis*

Suppose $T \in \mathcal{L}(V)$. A basis of V is called a *Jordan basis* for T if with respect to this basis T has a block diagonal matrix

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix},$$

where each A_j is of the form

$$A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}.$$

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