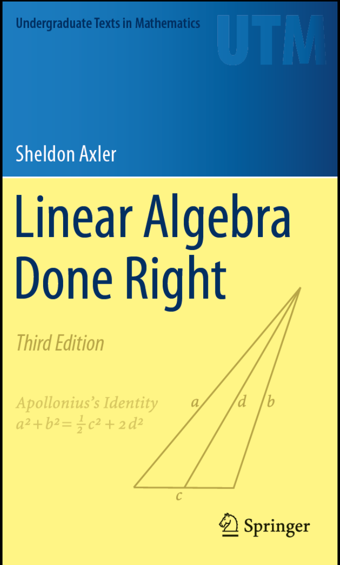
Invariant Subspaces



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- $\mathcal{L}(V) = \mathcal{L}(V, V)$

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Example: Suppose that $T \in \mathcal{L}\big(\mathcal{P}(\mathbf{R})\big)$ is defined by $Tp = p'$. Then $\mathcal{P}_4(\mathbf{R})$ is invariant under T because if $p \in \mathcal{P}(\mathbf{R})$ has degree at most 4, then p' also has degree at most 4.

Invariant Subspaces of Dimension 1

Suppose $v \in V$ and $v \neq 0$. Let

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Equivalent conditions to be an eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in F$. Then the following are equivalent:

- λ is an eigenvalue of T;
- $T - \lambda I$ is not injective;
- $T - \lambda I$ is not surjective;
- $T - \lambda I$ is not invertible.

An Operator with No Eigenvalues

Example: Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by

$$T(x, y) = (-y, x).$$

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A 90° counterclockwise rotation of a nonzero vector in $\mathbf{R}^2$ obviously never equals a scalar multiple of itself.

Conclusion: T has no eigenvalues.

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Also

$$\begin{aligned} T(1, i) &= (-i, 1) \\ &= -i(1, i). \end{aligned}$$

Thus $-i$ is an eigenvalue of T.

Eigenvectors

Definition: ***eigenvector***

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

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If $b \in \mathbf{C}$ and $b \neq 0$, then $(b, -bi)$ is also an eigenvector corresponding to the eigenvalue i because

$$T(b, -bi) = i(b, -bi).$$

Linearly Independent Eigenvectors

Linearly independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

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Proof   Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \ldots, v_m$ be corresponding eigenvectors. Then the list $v_1, \ldots, v_m$ is linearly independent. Thus $m \leq \dim V$, as desired. ■

Linear Algebra Done Right, by Sheldon Axler