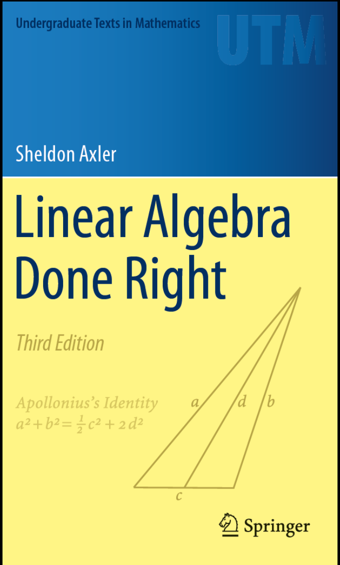
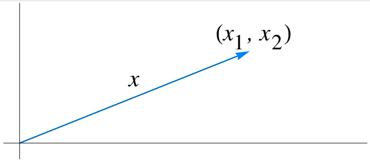
Inner Products and Norms, part 1: Inner Products



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a vector space over $\mathbf{F}$.

Dot Product on $\mathbf{R}^n$



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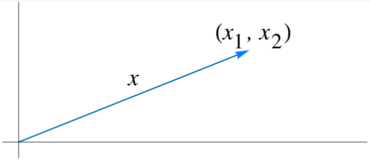

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For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, define the norm of x by $\|x\| = \sqrt{x_1{}^2 + \cdots + x_n{}^2}$.

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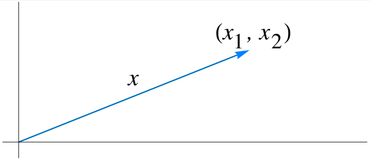

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For $x, y \in \mathbf{R}^n$, the *dot product* of x and y, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n,$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

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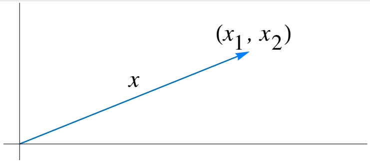

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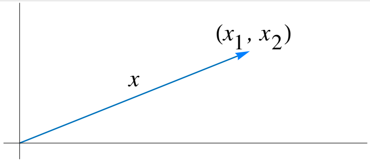

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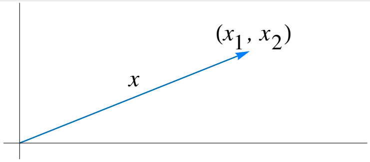

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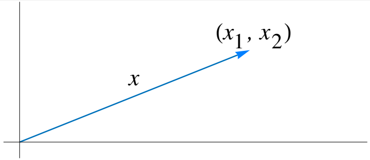

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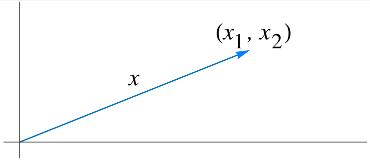

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- for $y \in \mathbf{R}^n$ fixed, the map from $\mathbf{R}^n$ to $\mathbf{R}$ that sends $x \in \mathbf{R}^n$ to $x \cdot y$ is linear;
- $x \cdot y = y \cdot x$ for all $x, y \in \mathbf{R}^n$.

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Recall that if $\lambda = a + bi$, where $a, b \in \mathbf{R}$, then

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For $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$, define the norm of z by

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This suggests that the inner product of $w = (w_1, \ldots, w_n) \in \mathbf{C}^n$ with z should equal

$$w_1\overline{z_1} + \cdots + w_n\overline{z_n}.$$

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Definition: *inner product*

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- homogeneity in first slot: $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$;
- conjugate symmetry: $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

Examples of Inner Products

- The *Euclidean inner product* on $\mathbf{F}^n$ is defined by
$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}.$$

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- $\langle u, \lambda v \rangle = \bar{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and $u, v \in V$.

Linear Algebra Done Right, by Sheldon Axler