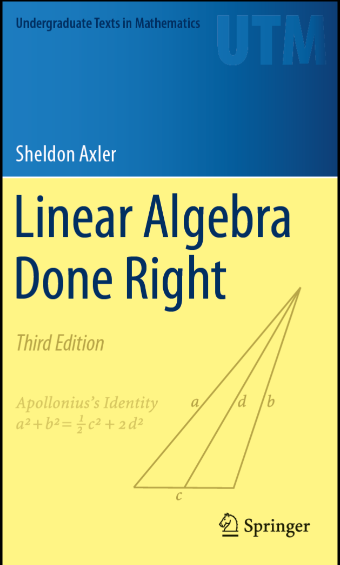
# Dimension


Undergraduate Texts in Mathematics
UTM
Sheldon Axler
Linear Algebra Done Right
Third Edition
Apollonius's Identity
$a^2+b^2=\frac{1}{2}c^2+2d^2$
a
d
b
c
Springer

# Review

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$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

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Example: The list $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbf{F}^n$, called the *standard basis* of $\mathbf{F}^n$.

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How should we define the dimension of a vector space?

# Basis Length is the Same for All Bases of $V$

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Interchanging the roles of $B_1$ and $B_2$, we also see that the length of $B_2$ is less than or equal to the length of $B_1$.

Thus the length of $B_1$ equals the length of $B_2$, as desired. ∎

# Definition of Dimension

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Examples:

- $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length $n$.
- $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

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However, every basis of $V$ has length $n$, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$.

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In other words, $v_1, \ldots, v_n$ is a basis of $V$, as desired. ∎

# Example Checking Just Linear Independence

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Proof

This list of two vectors in $\mathbf{F}^2$ is linearly independent because neither vector is a scalar multiple of the other.

Note that $\mathbf{F}^2$ has dimension $2$.

Thus this linearly independent list of length $2$ is a basis of $\mathbf{F}^2$ (we do not need to bother checking that it spans $\mathbf{F}^2$). ∎

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# Dimension of a Sum

### *Dimension of a sum*

If $U_1$ and $U_2$ are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

*Linear Algebra Done Right*, by Sheldon Axler