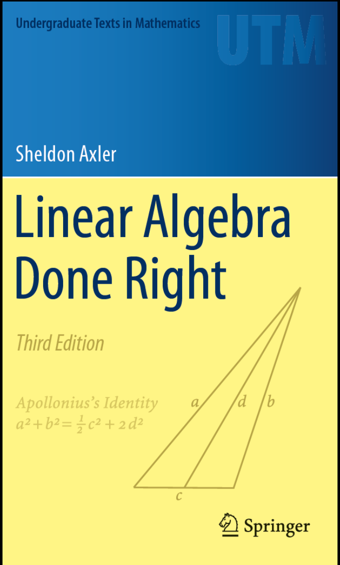
Determinant of an Operator and of a Matrix



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Definition of Determinant of an Operator

Definition: *determinant of an operator*, $\det T$

Suppose $T \in \mathcal{L}(V)$.

- If $\mathbf{F} = \mathbf{C}$, then the *determinant* of T is the product of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.
- If $\mathbf{F} = \mathbf{R}$, then the *determinant* of T is the product of the eigenvalues of $T_{\mathbf{C}}$, with each eigenvalue repeated according to its multiplicity.

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If $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T (or of $T_{\mathbf{C}}$ if V is a real vector space) with multiplicities $d_1, \ldots, d_m$, then the definition implies

$$\det T = \lambda_1^{d_1} \cdots \lambda_m^{d_m}.$$

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Or if you prefer to list the eigenvalues with each repeated according to its multiplicity, then the eigenvalues could be denoted $\lambda_1, \dots, \lambda_n$ (where the index n equals $\dim V$) and the definition implies

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Thus

$$\begin{aligned} \det T &= 1 \cdot (2+3i) \cdot (2-3i) \\ &= 13. \end{aligned}$$

Determinant and Characteristic Polynomial

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$$(z-\lambda_1)\cdots(z-\lambda_n).$$

Expand the polynomial above to write the characteristic polynomial of T in the form

$$z^n-(\lambda_1+\cdots+\lambda_n)z^{n-1}+\cdots+(-1)^n(\lambda_1\cdots\lambda_n).$$

The expression above gives the desired result. ∎

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Again, T is invertible if and only if 0 is not an eigenvalue of T, which happens if and only if 0 is not an eigenvalue of $T_{\mathbf{C}}$ (because $T_{\mathbf{C}}$ and T have the same real eigenvalues).

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Again, T is invertible if and only if 0 is not an eigenvalue of T, which happens if and only if 0 is not an eigenvalue of $T_{\mathbf{C}}$ (because $T_{\mathbf{C}}$ and T have the same real eigenvalues).

Thus again we see that T is invertible if and only if $\det T \neq 0$. ∎

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Proof First suppose $\mathbf{F} = \mathbf{C}$. If $\lambda, z \in \mathbf{C}$, then

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The right side of the equation above is, by definition, the characteristic polynomial of T, completing the proof when $\mathbf{F} = \mathbf{C}$.

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Now suppose $\mathbf{F} = \mathbf{R}$. Applying the complex case to $T_{\mathbf{C}}$ gives the desired result. ∎

Permutations

Definition: *permutation*, $\operatorname{perm} n$

- A *permutation* of $(1, \ldots, n)$ is a list $(m_1, \ldots, m_n)$ that contains each of the numbers $1, \ldots, n$ exactly once.
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The *sign* of a permutation $(m_1, \ldots, m_n)$ is defined to be 1 if the number of pairs of integers (j, k) with $1 \leq j < k \leq n$ such that j appears after k in the list $(m_1, \ldots, m_n)$ is even and -1 if the number of such pairs is odd.

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In other words, the sign of a permutation equals 1 if the natural order has been changed an even number of times and equals -1 if the natural order has been changed an odd number of times.

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Interchanging two entries in a permutation

Interchanging two entries in a permutation multiplies the sign of the permutation by -1.

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Suppose A is an n-by-n matrix

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix}.$$

The *determinant* of A, denoted $\det A$, is defined by

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The permutation $(1, 2, \dots, n)$ has sign 1 and thus contributes a term of $A_{1,1} \cdots A_{n,n}$ to the sum defining $\det A$.

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$$\det A = A_{1,1}\cdots A_{n,n}.$$

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Determinant of matrix of operator does not depend on basis

Let $T \in \mathcal{L}(V)$. Suppose $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Then

$$\det \mathcal{M}\big(T, (u_1, \dots, u_n)\big) = \det \mathcal{M}\big(T, (v_1, \dots, v_n)\big).$$

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Let $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

$$\det \mathcal{M}\big(T, (u_1, \ldots, u_n)\big) = \det \mathcal{M}\big(T, (v_1, \ldots, v_n)\big).$$

Proof Let

$A = \mathcal{M}\big(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big)$.

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The last equality completes the proof.

■

Determinant of Operator Equals Determinant of Its Matrix

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Suppose $T \in \mathcal{L}(V)$. Then $\det T = \det \mathcal{M}(T)$.

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Proof To show that $\det T = \det \mathcal{M}(T)$ for every basis of V, we need only show that the result holds for some basis of V.

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Interchanging S and T gives

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