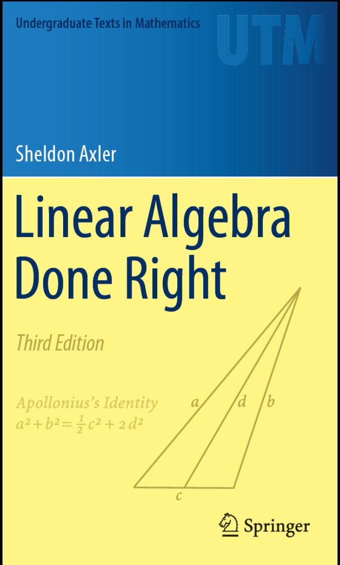
Decomposition via Generalized Eigenvectors



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Null Space and Range of $p(T)$ Are Invariant

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Decomposition Theorem

Description of operators on complex vector spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then

(a) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$;

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(c) each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent.

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Proof of (c): Clear from definition of $(T - \lambda_j I)|_{G(\lambda_j, T)}$.

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A basis of generalized eigenvectors

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Suppose that V is a complex vector space and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.

Proof Choose a basis of each $G(\lambda_j, T)$. Put all these bases together to form a basis of V consisting of generalized eigenvectors of T. ∎

Multiplicity

Definition: *multiplicity*

- Suppose $T \in \mathcal{L}(V)$. The *multiplicity* of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$.
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Proof $\quad V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

algebraic multiplicity of $\lambda = \dim \operatorname{null}(T - \lambda I)^{\dim V} = \dim G(\lambda, T)$

geometric multiplicity of $\lambda = \dim \operatorname{null}(T - \lambda I) = \dim E(\lambda, T)$

Multiplicity Example

Define $T \in \mathcal{L}(\mathbf{C}^3)$ by

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Thus the eigenvalue 3 has algebraic multiplicity 2 and the eigenvalue 8 has algebraic multiplicity 1.

We have

$$\mathbf{C}^3 = G(3, T) \oplus G(8, T),$$

as expected by the Decomposition Theorem.

Linear Algebra Done Right, by Sheldon Axler