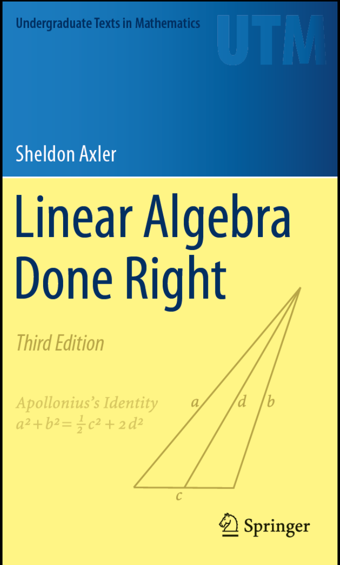
Change of Basis



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Identity Matrix; Inverse of a Matrix

Definition: *identity matrix, I*

Suppose n is a positive integer. The n-by-n diagonal matrix

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

is called the *identity matrix*.

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Definition: *invertible, inverse,* A^{-1}

A square matrix A is called *invertible* if there is a square matrix B of the same size such that

$$AB = BA = I;$$

we call B the *inverse* of A and denote it by A^{-1}.

Matrix of an Operator

Definition: ***matrix of an operator,*** $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are bases of V. The *matrix of* T with respect to these bases is the n-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{n,k}w_n.$$

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If the bases are not clear from the context, then use the notation $\mathcal{M}\big(T, (v_1, \ldots, v_n), (w_1, \ldots, w_n)\big)$.

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If the bases are not clear from the context, then use the notation $\mathcal{M}\big(T, (v_1, \ldots, v_n), (w_1, \ldots, w_n)\big)$.

$$\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_n \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \ldots & v_k & \ldots & v_n \end{array} \\ \left(\begin{array}{ccccc} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{n,k} & & \end{array} \right) \end{array}.$$

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The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_n$:

$$Tv_k = \sum_{j=1}^{n} A_{j,k}w_j.$$

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$$\mathcal{M}\big(I, (v_1, \dots, v_n), (v_1, \dots, v_n)\big) = I$$

Matrix of the Identity with Respect to Two Bases

The matrix of the product of operators

$$\mathcal{M}\big(ST, (u_1, \ldots, u_n), (w_1, \ldots, w_n)\big) =$$
$$\mathcal{M}\big(S, (v_1, \ldots, v_n), (w_1, \ldots, w_n)\big)\mathcal{M}\big(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big).$$

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Suppose $u_1, \dots, u_n$ and $v_1, \dots, v_n$ are bases of V. Then

$$\mathcal{M}\big(I, (u_1, \dots, u_n), (v_1, \dots, v_n)\big)$$

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are inverses of each other.

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$$I = \mathcal{M}\big(I, (v_1, \dots, v_n), (u_1, \dots, u_n)\big) \cdot \mathcal{M}\big(I, (u_1, \dots, u_n), (v_1, \dots, v_n)\big).$$

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Now interchange the roles of the u's and v's, getting the product in the other order. ∎

Change of Basis Formula

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Suppose $T \in \mathcal{L}(V)$. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be bases of V. Let

$$A = \mathcal{M}\big(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n)\big).$$

Then

$$\mathcal{M}\big(T, (u_1, \ldots, u_n)\big) = A^{-1}\mathcal{M}\big(T, (v_1, \ldots, v_n)\big)A.$$

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Now in result above replace w_j with v_j.

Change of Basis Formula

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Suppose $T \in \mathcal{L}(V)$. Let $u_1,\dots,u_n$ and $v_1,\dots,v_n$ be bases of V. Let

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Now in result above replace w_j with v_j. Also replace T with I

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Change of Basis Formula

The matrix of the product of operators

$$\mathcal{M}\big(T, (u_1, \dots, u_n), (v_1, \dots, v_n)\big) =$$
$$\mathcal{M}\big(T, (v_1, \dots, v_n), (v_1, \dots, v_n)\big)\mathcal{M}\big(I, (u_1, \dots, u_n), (v_1, \dots, v_n)\big).$$

Change of basis formula

Suppose $T \in \mathcal{L}(V)$. Let $u_1, \dots, u_n$ and $v_1, \dots, v_n$ be bases of V. Let

$$A = \mathcal{M}\big(I, (u_1, \dots, u_n), (v_1, \dots, v_n)\big).$$

Then

$$\mathcal{M}\big(T, (u_1, \dots, u_n)\big) = A^{-1}\mathcal{M}\big(T, (v_1, \dots, v_n)\big)A.$$

Proof In the result above, replace w_j with u_j and replace S with I, getting

$$\mathcal{M}\big(T, (u_1, \dots, u_n)\big) = A^{-1}\mathcal{M}\big(T, (u_1, \dots, u_n), (v_1, \dots, v_n)\big).$$

Now in result above replace w_j with v_j. Also replace T with I and replace S with T, getting

$$\mathcal{M}\big(T, (u_1, \dots, u_n), (v_1, \dots, v_n)\big) = \mathcal{M}\big(T, (v_1, \dots, v_n)\big)A.$$

Change of Basis Formula

The matrix of the product of operators

$$\mathcal{M}(T,(u_1,\dots,u_n),(v_1,\dots,v_n)) =$$
$$\mathcal{M}(T,(v_1,\dots,v_n),(v_1,\dots,v_n))\mathcal{M}(I,(u_1,\dots,u_n),(v_1,\dots,v_n)).$$

Change of basis formula

Suppose $T \in \mathcal{L}(V)$. Let $u_1,\dots,u_n$ and $v_1,\dots,v_n$ be bases of V. Let

$$A = \mathcal{M}(I,(u_1,\dots,u_n),(v_1,\dots,v_n)).$$

Then

$$\mathcal{M}(T,(u_1,\dots,u_n)) = A^{-1}\mathcal{M}(T,(v_1,\dots,v_n))A.$$

Proof In the result above, replace w_j with u_j and replace S with I, getting

$$\mathcal{M}(T,(u_1,\dots,u_n)) = A^{-1}\mathcal{M}(T,(u_1,\dots,u_n),(v_1,\dots,v_n)).$$

Now in result above replace w_j with v_j. Also replace T with I and replace S with T, getting

$$\mathcal{M}(T,(u_1,\dots,u_n),(v_1,\dots,v_n)) = \mathcal{M}(T,(v_1,\dots,v_n))A.$$

Substitute, getting the desired result. ∎

Linear Algebra Done Right, by Sheldon Axler