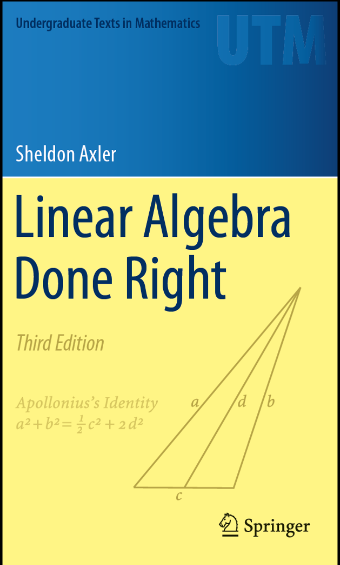
Bases


Undergraduate Texts in Mathematics
UTM
Sheldon Axler
Linear Algebra Done Right
Third Edition
Apollonius's Identity
$a^2+b^2=\frac{1}{2}c^2+2d^2$
a
d
b
c
Springer

Notation

Reminder:

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V denotes a vector space over $\mathbf{F}$.

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- $(3 + 2i)z^2 + 4iz + 9 \in \mathcal{P}_{20}(\mathbf{C})$

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- The list $(1,1,0),(0,0,1)$ is a basis of $\{(x,x,y)\in\mathbf{F}^3 : x,y\in\mathbf{F}\}$.
- The list $(1,2),(3,5)$ is a basis of $\mathbf{F}^2$.
- The list $(1,-1,0),(1,0,-1)$ is a basis of
$$\{(x,y,z)\in\mathbf{F}^3 : x+y+z=0\}.$$
- The list $1,z,\ldots,z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

Non-examples of Bases

- The list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$.

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- The list $(1, 2), (3, 5), (4, 13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent.

Why Bases Are Useful

Basis gives unique representation as linear combination

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

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The list $(1, -1, 0), (1, 0, -1)$ is a basis of V.

If $(x, y, z) \in V$, then

$$(x, y, z) = -y(1, -1, 0) + (-z)(1, 0, -1).$$

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Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Every linearly independent list extends to a basis

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Linear Algebra Done Right, by Sheldon Axler