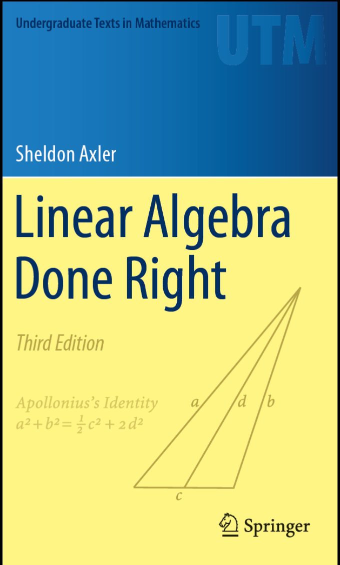
Duality, part 2: Annihilators and the Matrix of a Dual Map



Notation

Definition: *annihilator,* U^0

For $U \subset V$, the *annihilator* of U, denoted U^0, is defined by

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Dimension of the annihilator

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim U + \dim U^0 = \dim V.$$

The Null Space and Range of a Dual Map

The null space and range of T'

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- $\operatorname{null} T' = (\operatorname{range} T)^0$;
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T surjective is equivalent to T' injective

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- T is surjective if and only if T' is injective;
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The Transpose of a Matrix

Definition: *transpose*, A^{t}

The *transpose* of a matrix A, denoted A^{t}, is the matrix obtained from A by interchanging the rows and columns. More specifically, if A is an m-by-n matrix, then A^{t} is the n-by-m matrix whose entries are given by the equation

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Example: If $A = \begin{pmatrix} 5 & -7 \\ 3 & 8 \\ -4 & 2 \end{pmatrix}$,

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The transpose of a product

If A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^{\mathrm{t}} = C^{\mathrm{t}}A^{\mathrm{t}}.$$

The Matrix of the Dual of a Linear Map

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The matrix of T' is the transpose of the matrix of T

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

$$\mathcal{M}(T') = \big(\mathcal{M}(T)\big)^{\mathrm{t}}.$$

Row Rank and Column Rank

Definition: *row rank, column rank*

Suppose A is an m-by-n matrix with entries in $\mathbf{F}$.

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in $\mathbf{F}^{1,4}$.

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Column Rank Equals Dimension of Range

Dimension of $\operatorname{range} T$ ***equals column rank of*** $\mathcal{M}(T)$

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim \operatorname{range} T$ equals the column rank of $\mathcal{M}(T)$.

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Definition: ***rank***

The *rank* of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.

Linear Algebra Done Right, by Sheldon Axler