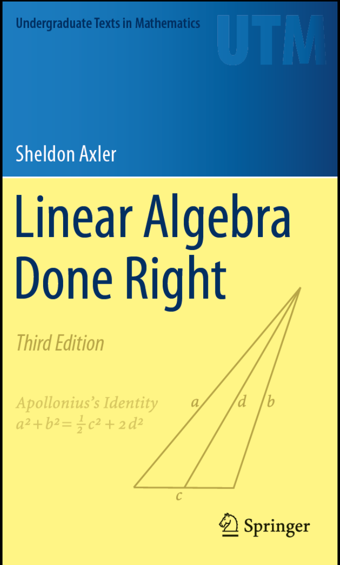
Upper-Triangular Matrices



The Matrix of an Operator

Definition: *matrix of an operator,* $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of T* with respect to this basis is the n-by-n matrix

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

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- $\mathcal{M}(T)$ is a ***square*** matrix.

Example: Define $T \in \mathcal{L}(\mathbf{R}^3)$ by

$$T(x, y, z) = (2x+y, 5y+3z, 8z).$$

Then

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

with respect to the standard basis of $\mathbf{R}^3$.

Upper-Triangular Matrices

Example:

$$\begin{pmatrix} \color{red}{2} & 1 & 0 \\ 0 & \color{red}{5} & 3 \\ 0 & 0 & \color{red}{8} \end{pmatrix}$$

Definition: *diagonal of matrix*

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Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent:

- the matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular;
- $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ for each $j = 1, \ldots, n$;
- $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under T for each $j = 1, \ldots, n$.

Over $\mathbf{C}$, Every Operator has Upper-Triangular Matrix

Over $\mathbf{C}$, every operator has an upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

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The first basis vector v_1 must be an eigenvector of T with eigenvalue λ_1.

Eigenvalues Along the Diagonal

Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

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Thus the eigenvalues of T are 2, 5, and 8.

Linear Algebra Done Right, by Sheldon Axler