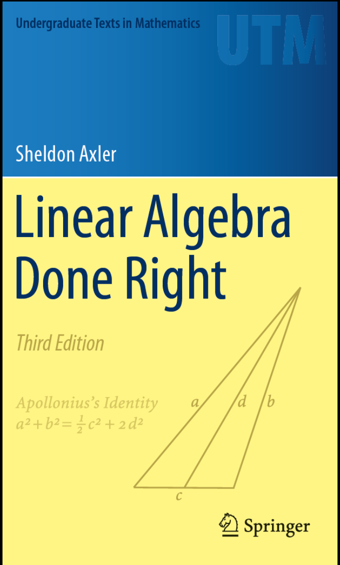
Trace of an Operator and of a Matrix



Review of Multiplicity

Definition: *Multiplicity*

Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. Then

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Definition: *Repeated accord to multiplicity*

Suppose $T \in \mathcal{L}(V)$. The phrase '*with each eigenvalue repeated according to its multiplicity* means that if $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T with multiplicities $d_1, \ldots, d_m$, then we create a list with λ_1 listed d_1 times, ..., λ_m listed d_m times. If $\mathbf{F} = \mathbf{C}$, then this list has length $\dim V$.

Trace of an Operator

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Suppose $T \in \mathcal{L}(V)$.

- If $\mathbf{F} = \mathbf{C}$, then the *trace* of T is the sum of the eigenvalues of T, with each eigenvalue repeated according to its multiplicity.
- If $\mathbf{F} = \mathbf{R}$, then the *trace* of T is the sum of the eigenvalues of $T_{\mathbf{C}}$, with each eigenvalue repeated according to its multiplicity.

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$$\begin{aligned} \operatorname{trace} T &= 1 + (2+3i) + (2-3i) \\ &= 5. \end{aligned}$$

Trace and characteristic polynomial

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $\operatorname{trace} T$ equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.

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Proof Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T (or of $T_{\mathbf{C}}$ if V is a real vector space) with each eigenvalue repeated according to its multiplicity. Then the characteristic polynomial of T equals

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Expand the polynomial above to write the characteristic polynomial of T in the form

$$z^n - (\lambda_1 + \cdots + \lambda_n) z^{n-1} + \cdots + (-1)^n (\lambda_1 \cdots \lambda_n).$$

The expression above gives the desired result. ■

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Trace of AB equals trace of BA

If A and B are square matrices of the same size, then

$$\operatorname{trace}(AB) = \operatorname{trace}(BA).$$

Trace Does Not Depend on Basis

Trace of matrix of operator does not depend on basis

Let $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then

$$\operatorname{trace} \mathcal{M}\big(T, (u_1, \ldots, u_n)\big) = \operatorname{trace} \mathcal{M}\big(T, (v_1, \ldots, v_n)\big).$$

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The last equality completes the proof. ∎

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If V is a real vector space, then applying the complex case to the complexification $T_{\mathbf{C}}$ (which is used to define $\operatorname{trace} T$) gives the desired result.

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completing the proof. ∎

Exercises

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- Suppose V is an inner product space. Prove that
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defines an inner product on $\mathcal{L}(V)$.
- Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of T, repeated according to multiplicity. Suppose
$$\begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix}$$
is the matrix of T with respect to some orthonormal basis of V. Prove that
$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \le \sum_{k=1}^{n} \sum_{j=1}^{n} |A_{j,k}|^2.$$

Linear Algebra Done Right, by Sheldon Axler