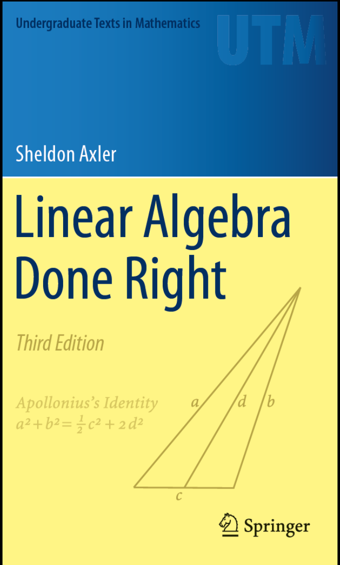
The Spectral Theorem



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional inner product space over $\mathbf{F}$.

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

- An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of the operator.

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
- An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of the operator.
- For which operators on V is there an *orthonormal* basis of V with respect to which the operator has a diagonal matrix?

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
- An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of the operator.
- For which operators on V is there an *orthonormal* basis of V with respect to which the operator has a diagonal matrix?
- For which operators on V is there an *orthonormal* basis of V consisting of eigenvectors of T?

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
- An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of the operator.
- For which operators on V is there an *orthonormal* basis of V with respect to which the operator has a diagonal matrix?
- For which operators on V is there an *orthonormal* basis of V consisting of eigenvectors of T?
- The Spectral Theorem will answer these questions.

Diagonalization by an Orthonormal Basis?

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
- An operator on V has a diagonal matrix with respect to a basis if and only if the basis consists of eigenvectors of the operator.
- For which operators on V is there an *orthonormal* basis of V with respect to which the operator has a diagonal matrix?
- For which operators on V is there an *orthonormal* basis of V consisting of eigenvectors of T?
- The Spectral Theorem will answer these questions.
- The answer is depends upon whether the scalar field is real or complex.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The matrix of T^* is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The matrix of T^* is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix.

Any two diagonal matrices commute; thus T commutes with T^*.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The matrix of T^* is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix.

Any two diagonal matrices commute; thus T commutes with T^*.

Thus T is normal. In other words, (a) holds.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The matrix of T^* is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix.

Any two diagonal matrices commute; thus T commutes with T^*.

Thus T is normal. In other words, (a) holds.

We have proved that (c) implies (a). The equivalence of (b) and (c) is easy.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof First suppose (c) holds.

The matrix of T^* is obtained by taking the conjugate transpose of the matrix of T; hence T^* also has a diagonal matrix.

Any two diagonal matrices commute; thus T commutes with T^*.

Thus T is normal. In other words, (a) holds.

We have proved that (c) implies (a). The equivalence of (b) and (c) is easy.

We will complete the proof by showing that (a) implies (c).

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof Suppose (a) holds, so T is normal.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof Suppose (a) holds, so T is normal.
By Schur's Theorem , there is an orthonormal basis $e_1, \ldots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus

$$\mathcal{M}\big(T, (e_1, \ldots, e_n)\big) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof Suppose (a) holds, so T is normal. By Schur's Theorem , there is an orthonormal basis $e_1, \ldots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus

$$\mathcal{M}(T, (e_1, \ldots, e_n)) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$

The condition $\|Te_1\| = \|T^*e_1\|$ implies that all entries in the first row of the matrix, except possibly the first entry $a_{1,1}$, equal 0.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof Suppose (a) holds, so T is normal. By Schur's Theorem , there is an orthonormal basis $e_1, \ldots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus

$$\mathcal{M}(T, (e_1, \ldots, e_n)) = \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$

The condition $\|Te_1\| = \|T^*e_1\|$ implies that all entries in the first row of the matrix, except possibly the first entry $a_{1,1}$, equal 0.

The condition $\|Te_2\| = \|T^*e_2\|$ implies that all entries in the second row of the matrix, except possibly the diagonal entry $a_{2,2}$, equal 0.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof Suppose (a) holds, so T is normal. By Schur's Theorem , there is an orthonormal basis $e_1, \dots, e_n$ of V with respect to which T has an upper-triangular matrix. Thus

$$\mathcal{M}\big(T, (e_1, \dots, e_n)\big) = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{pmatrix}.$$

The condition $\|Te_1\| = \|T^*e_1\|$ implies that all entries in the first row of the matrix, except possibly the first entry $a_{1,1}$, equal 0.

The condition $\|Te_2\| = \|T^*e_2\|$ implies that all entries in the second row of the matrix, except possibly the diagonal entry $a_{2,2}$, equal 0.

Continuing in this fashion, we see that all the nondiagonal entries in the matrix equal 0. Thus the matrix above is diagonal, and (c) holds. ■

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the normal operator $T \in \mathcal{L}(\mathbf{C}^2)$ whose matrix is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the normal operator $T \in \mathcal{L}(\mathbf{C}^2)$ whose matrix is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

As you can verify,

$$\frac{(i,1)}{\sqrt{2}}, \frac{(-i,1)}{\sqrt{2}}$$

is an orthonormal basis of $\mathbf{C}^2$ consisting of eigenvectors of T.

The Complex Spectral Theorem

Complex Spectral Theorem

Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is normal.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the normal operator $T \in \mathcal{L}(\mathbf{C}^2)$ whose matrix is

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}.$$

As you can verify,

$$\frac{(i,1)}{\sqrt{2}}, \frac{(-i,1)}{\sqrt{2}}$$

is an orthonormal basis of $\mathbf{C}^2$ consisting of eigenvectors of T.

With respect to this basis the matrix of T is the diagonal matrix

$$\begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}.$$

The Real Spectral Theorem

Real Spectral Theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Real Spectral Theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the self-adjoint operator T on $\mathbf{R}^3$ whose matrix is

$$\begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}.$$

The Real Spectral Theorem

Real Spectral Theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the self-adjoint operator T on $\mathbf{R}^3$ whose matrix is

$$\begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}.$$

As you can verify,

$$\frac{(1,-1,0)}{\sqrt{2}}, \frac{(1,1,1)}{\sqrt{3}}, \frac{(1,1,-2)}{\sqrt{6}}$$

is an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors of T.

The Real Spectral Theorem

Real Spectral Theorem

Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Example: Consider the self-adjoint operator T on $\mathbf{R}^3$ whose matrix is

$$\begin{pmatrix} 14 & -13 & 8 \\ -13 & 14 & 8 \\ 8 & 8 & -7 \end{pmatrix}.$$

As you can verify,

$$\frac{(1,-1,0)}{\sqrt{2}}, \frac{(1,1,1)}{\sqrt{3}}, \frac{(1,1,-2)}{\sqrt{6}}$$

is an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors of T. With respect to this basis, the matrix of T is the diagonal matrix

$$\begin{pmatrix} 27 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -15 \end{pmatrix}.$$

Linear Algebra Done Right, by Sheldon Axler