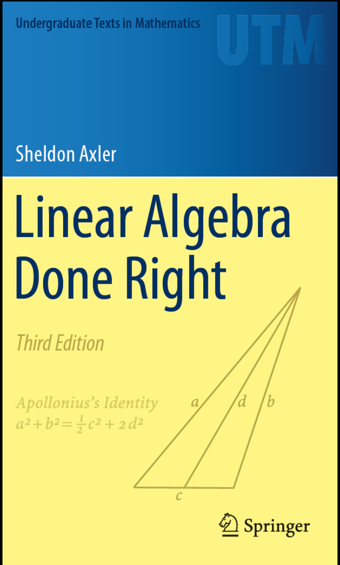
Polar Decomposition and SVD, part 2: Singular Value Decomposition



Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Example:Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Thus

$$\sqrt{T^*T}(z_1, z_2, z_3, z_4) = (3z_1, 2z_2, 0, 3z_4).$$

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Thus

$$\sqrt{T^*T}(z_1, z_2, z_3, z_4) = (3z_1, 2z_2, 0, 3z_4).$$

The eigenvalues of $\sqrt{T^*T}$ are $3, 2, 0$ and

$$\dim \operatorname{null}(\sqrt{T^*T} - 3I) = 2,$$

$$\dim \operatorname{null}(\sqrt{T^*T} - 2I) = 1,$$

$$\dim \operatorname{null} \sqrt{T^*T} = 1.$$

Singular Values

Definition: *singular values*

Suppose $T \in \mathcal{L}(V)$. The *singular values* of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ listed

$$\dim \operatorname{null}(\sqrt{T^*T} - \lambda I)$$

times.

The singular values of T are nonnegative numbers, because they are eigenvalues of the positive operator $\sqrt{T^*T}$.

Each operator on V has $\dim V$ singular values.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Thus

$$\sqrt{T^*T}(z_1, z_2, z_3, z_4) = (3z_1, 2z_2, 0, 3z_4).$$

The eigenvalues of $\sqrt{T^*T}$ are $3, 2, 0$ and

$$\dim \operatorname{null}(\sqrt{T^*T} - 3I) = 2,$$
$$\dim \operatorname{null}(\sqrt{T^*T} - 2I) = 1,$$
$$\dim \operatorname{null} \sqrt{T^*T} = 1.$$

Hence the singular values of T are $3, 3, 2, 0$.

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

With notation as in the statement of the Singular Value Decomposition,

$$Te_j = s_j f_j$$

for each j.

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

With notation as in the statement of the Singular Value Decomposition,

$$Te_j = s_j f_j$$

for each j. Thus

$$\mathcal{M}\big(T, (e_1, \ldots, e_n), (f_1, \ldots, f_n)\big) = \begin{pmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{pmatrix}.$$

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\sqrt{T^*T}e_j = s_je_j$ for $j = 1, \ldots, n$.

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\sqrt{T^*T}e_j = s_j e_j$ for $j = 1, \ldots, n$.

Now

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

for every $v \in V$.

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\sqrt{T^*T} e_j = s_j e_j$ for $j = 1, \ldots, n$. Now

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

for every $v \in V$.

Apply $\sqrt{T^*T}$ to both sides of this equation, getting

$$\sqrt{T^*T} v = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n$$

for every $v \in V$.

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\sqrt{T^*T}e_j = s_je_j$ for $j = 1, \ldots, n$. Now

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

for every $v \in V$.

Apply $\sqrt{T^*T}$ to both sides of this equation, getting

$$\sqrt{T^*T}v = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n$$

for every $v \in V$. By the Polar Decomposition, there is an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \ldots, s_n$. Then there exist orthonormal bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V such that $\sqrt{T^*T} e_j = s_j e_j$ for $j = 1, \ldots, n$. Now

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

for every $v \in V$.

Apply $\sqrt{T^*T}$ to both sides of this equation, getting

$$\sqrt{T^*T} v = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n$$

for every $v \in V$. By the Polar Decomposition, there is an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$. Apply S to both sides of the equation above, getting

$$Tv = s_1 \langle v, e_1 \rangle Se_1 + \cdots + s_n \langle v, e_n \rangle Se_n$$

for every $v \in V$.

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \dots, s_n$. Then there exist orthonormal bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of V such that

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \dots, e_n$ of V such that $\sqrt{T^*T}e_j = s_je_j$ for $j = 1, \dots, n$. Now

$$v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_n\rangle e_n$$

for every $v \in V$.

Apply $\sqrt{T^*T}$ to both sides of this equation, getting

$$\sqrt{T^*T}v = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n$$

for every $v \in V$. By the Polar Decomposition, there is an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$. Apply S to both sides of the equation above, getting

$$Tv = s_1\langle v, e_1\rangle Se_1 + \cdots + s_n\langle v, e_n\rangle Se_n$$

for every $v \in V$. For each j, let $f_j = Se_j$. Because S is an isometry, $f_1, \dots, f_n$ is an orthonormal basis of V.

Singular Value Decomposition

Singular Value Decomposition

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, \dots, s_n$. Then there exist orthonormal bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ of V such that

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$.

Proof By the Spectral Theorem, there is an orthonormal basis $e_1, \dots, e_n$ of V such that $\sqrt{T^*T} e_j = s_j e_j$ for $j = 1, \dots, n$. Now

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

for every $v \in V$.

Apply $\sqrt{T^*T}$ to both sides of this equation, getting

$$\sqrt{T^*T} v = s_1 \langle v, e_1 \rangle e_1 + \cdots + s_n \langle v, e_n \rangle e_n$$

for every $v \in V$. By the Polar Decomposition, there is an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$. Apply S to both sides of the equation above, getting

$$Tv = s_1 \langle v, e_1 \rangle S e_1 + \cdots + s_n \langle v, e_n \rangle S e_n$$

for every $v \in V$. For each j, let $f_j = S e_j$. Because S is an isometry, $f_1, \dots, f_n$ is an orthonormal basis of V. The equation above now becomes

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$, completing the proof. ∎

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example:Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

The eigenvalues of T^*T are $9, 4, 0$ and

$$\dim \operatorname{null}(T^*T - 9I) = 2,$$

$$\dim \operatorname{null}(T^*T - 4I) = 1,$$

$$\dim \operatorname{null} T^*T = 1.$$

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

The eigenvalues of T^*T are $9, 4, 0$ and

$$\dim \operatorname{null}(T^*T - 9I) = 2,$$
$$\dim \operatorname{null}(T^*T - 4I) = 1,$$
$$\dim \operatorname{null} T^*T = 1.$$

Hence the singular values of T are $3, 3, 2, 0$.

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

The eigenvalues of T^*T are $9, 4, 0$ and

$$\dim \operatorname{null}(T^*T - 9I) = 2,$$
$$\dim \operatorname{null}(T^*T - 4I) = 1,$$
$$\dim \operatorname{null} T^*T = 1.$$

Hence the singular values of T are $3, 3, 2, 0$.

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

The eigenvalues of T^*T are $9, 4, 0$ and

$$\dim \operatorname{null}(T^*T - 9I) = 2,$$
$$\dim \operatorname{null}(T^*T - 4I) = 1,$$
$$\dim \operatorname{null} T^*T = 1.$$

Hence the singular values of T are $3, 3, 2, 0$.

Singular Values Without Taking Square Roots

Singular values without taking square root of an operator

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue λ listed

$$\dim \operatorname{null}(T^*T - \lambda I)$$

times.

Example: Define $T \in \mathcal{L}(\mathbf{F}^4)$ by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4).$$

A calculation shows that

$$T^*T(z_1, z_2, z_3, z_4) = (9z_1, 4z_2, 0, 9z_4).$$

The eigenvalues of T^*T are $9, 4, 0$ and

$$\dim \operatorname{null}(T^*T - 9I) = 2,$$
$$\dim \operatorname{null}(T^*T - 4I) = 1,$$
$$\dim \operatorname{null} T^*T = 1.$$

Hence the singular values of T are $3, 3, 2, 0$.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
- Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
- Suppose $T \in \mathcal{L}(V)$. Prove that $\operatorname{dim} \operatorname{range} T$ equals the number of nonzero singular values of T.
- Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
- Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.
- Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.
- Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
- Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.
- Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.
- Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T.
  - (a) Prove that $\hat{s}\|v\| \leq \|Tv\| \leq s\|v\|$ for every $v \in V$.

Exercises

- Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that the singular values of T equal the absolute values of the eigenvalues of T, repeated appropriately.
- Suppose $T \in \mathcal{L}(V)$. Prove that T and T^* have the same singular values.
- Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.
- Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.
- Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.
- Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T.
  - (a) Prove that $\hat{s}\|v\| \leq \|Tv\| \leq s\|v\|$ for every $v \in V$.
  - (b) Suppose λ is an eigenvalue of T. Prove that $\hat{s} \leq |\lambda| \leq s$.

Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

Exercises

Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

- Prove that if $v \in V$, then

  $$T^*v = s_1\langle v, f_1\rangle e_1 + \cdots + s_n\langle v, f_n\rangle e_n.$$

Exercises

Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1 \langle v, e_1 \rangle f_1 + \cdots + s_n \langle v, e_n \rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

- Prove that if $v \in V$, then
$$T^* v = s_1 \langle v, f_1 \rangle e_1 + \cdots + s_n \langle v, f_n \rangle e_n.$$
- Prove that if $v \in V$, then
$$T^* T v = {s_1}^2 \langle v, e_1 \rangle e_1 + \cdots + {s_n}^2 \langle v, e_n \rangle e_n.$$

Exercises

Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

- Prove that if $v \in V$, then

  $$T^*v = s_1\langle v, f_1\rangle e_1 + \cdots + s_n\langle v, f_n\rangle e_n.$$

- Prove that if $v \in V$, then

  $$T^*Tv = {s_1}^2\langle v, e_1\rangle e_1 + \cdots + {s_n}^2\langle v, e_n\rangle e_n.$$

- Prove that if $v \in V$, then

  $$\sqrt{T^*T}v = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n.$$

Exercises

Suppose $T \in \mathcal{L}(V)$ has singular value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are orthonormal bases of V.

- Prove that if $v \in V$, then

  $$T^*v = s_1\langle v, f_1\rangle e_1 + \cdots + s_n\langle v, f_n\rangle e_n.$$

- Prove that if $v \in V$, then

  $$T^*Tv = {s_1}^2\langle v, e_1\rangle e_1 + \cdots + {s_n}^2\langle v, e_n\rangle e_n.$$

- Prove that if $v \in V$, then

  $$\sqrt{T^*T}v = s_1\langle v, e_1\rangle e_1 + \cdots + s_n\langle v, e_n\rangle e_n.$$

- Suppose T is invertible. Prove that if $v \in V$, then

  $$T^{-1}v = \frac{\langle v, f_1\rangle e_1}{s_1} + \cdots + \frac{\langle v, f_n\rangle e_n}{s_n}$$

  for every $v \in V$.

Linear Algebra Done Right, by Sheldon Axler