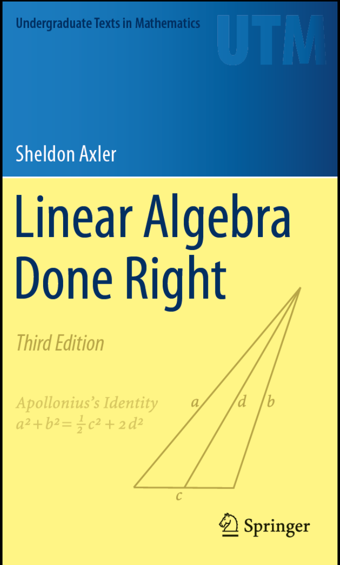
Products and Quotients of Vector Spaces, part 2: Quotients



The Sum of a Vector and a Subspace

Definition: $v + U$

Suppose $v \in V$ and U is a subspace of V. Then $v + U$ is the subset of V defined by

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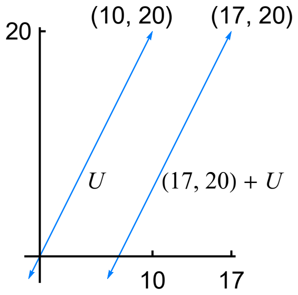
Then U is the line in $\mathbf{R}^2$ through the origin with slope 2.



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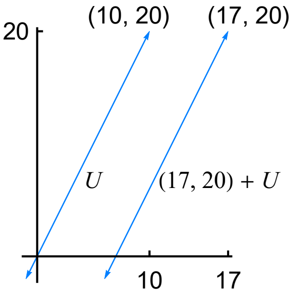
Then U is the line in $\mathbf{R}^2$ through the origin with slope 2. Thus

$$(17, 20) + U$$

is the line in $\mathbf{R}^2$ that contains the point $(17, 20)$ and has slope 2.



Affine Subsets

Definition: *affine subset, parallel*

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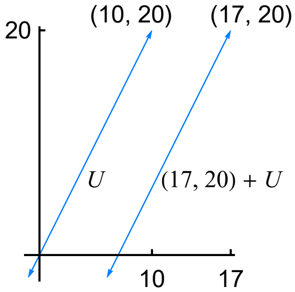

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If $U = \{(x, 2x) \in \mathbf{R}^2 : x \in \mathbf{R}\}$, then all the lines in $\mathbf{R}^2$ with slope 2 are parallel to U.

If $U = \{(x, y, 0) \in \mathbf{R}^3 : x, y \in \mathbf{R}\}$, then the affine subsets of $\mathbf{R}^3$ parallel to U are the planes in $\mathbf{R}^3$ that are parallel to the xy-plane U in the usual sense.

Quotient Spaces

Definition: *quotient space,* V/U

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$$V/U = \{v + U : v \in V\}.$$

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- If U is a line in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all lines in $\mathbf{R}^3$ parallel to U.
- If U is a plane in $\mathbf{R}^3$ containing the origin, then $\mathbf{R}^3/U$ is the set of all planes in $\mathbf{R}^3$ parallel to U.

Parallel Affine Subsets Are Either Equal or Disjoint

Two affine subsets parallel to U are equal or disjoint

Suppose U is a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v - w \in U$;

(b) $v + U = w + U$;

(c) $(v + U) \cap (w + U) \neq \varnothing$.

Making a Quotient Space a Vector Space

Definition: *addition and scalar multiplication on V/U*

Suppose U is a subspace of V. Then *addition* and *scalar multiplication* are defined on V/U by

$$(v+U)+(w+U)=(v+w)+U$$
$$\lambda(v+U)=(\lambda v)+U$$

for $v, w \in V$ and $\lambda \in \mathbf{F}$.

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Quotient space is a vector space

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

Quotient Map

Definition: *quotient map*, π

Suppose U is a subspace of V. The *quotient map* π is the linear map $\pi\colon V \to V/U$ defined by

$$\pi(v) = v + U$$

for $v \in V$.

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Dimension of a quotient space

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim V/U = \dim V - \dim U.$$

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Proof

$$\begin{aligned}\dim V &= \dim \operatorname{null} \pi + \dim \operatorname{range} \pi \\ &= \dim U + \dim V/U\end{aligned}$$

The Induced Map on $V/(\operatorname{null} T)$

Definition: $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T}\colon V/(\operatorname{null} T) \to W$ by

$$\tilde{T}(v + \operatorname{null} T) = Tv.$$

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Null space and range of $\tilde{T}$

Suppose $T \in \mathcal{L}(V, W)$. Then

(a) $\tilde{T}$ is a linear map from $V/(\operatorname{null} T)$ to W;

(b) $\tilde{T}$ is injective;

(c) $\operatorname{range} \tilde{T} = \operatorname{range} T$;

(d) $V/(\operatorname{null} T)$ is isomorphic to $\operatorname{range} T$.

Linear Algebra Done Right, by Sheldon Axler