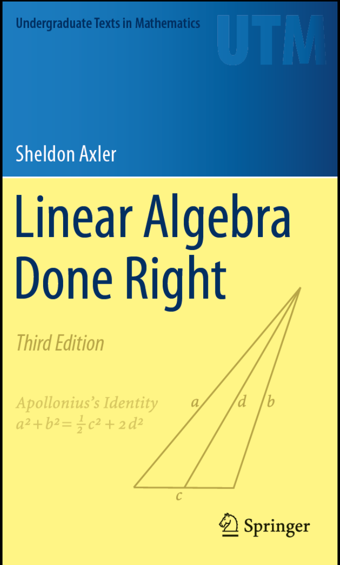
# Positive Operators and Isometries, part 1: Positive Operators



# Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- $V$ denotes a finite-dimensional inner product space over $\mathbf{F}$.

# Definition and Examples of Positive Operators

**Definition: *positive operator***

An operator $T \in \mathcal{L}(V)$ is called *positive* if $T$ is self-adjoint and

$$\langle Tv, v \rangle \geq 0$$

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- If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbf{R}$ are such that $b^2 \leq 4c$, then $T^2 + bT + cI$ is a positive operator.
- If $\mathcal{M}(T)$ is a diagonal matrix with nonnegative entries on the diagonal, then $T$ is a positive operator.

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Example: If $T \in \mathcal{L}(\mathbf{F}^3)$ is defined by

$$T(z_1, z_2, z_3) = (z_3, 0, 0),$$

then the operator $R \in \mathcal{L}(\mathbf{F}^3)$ defined by

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Example: If $T \in \mathcal{L}(\mathbf{F}^3)$ has matrix

$$\mathcal{M}(T) = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix},$$

then the operator $R \in \mathcal{L}(\mathbf{F}^3)$ with matrix

$$\mathcal{M}(R) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

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Thus $\lambda$ is a nonnegative number. Hence (b) holds.

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$$Re_j = \sqrt{\lambda_j} e_j$$

for $j = 1, \ldots, n$. Then $R$ is a positive operator.

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Thus $R$ is a positive square root of $T$. Hence (c) holds.

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A positive operator can have infinitely many square roots, although only one of them can be positive. For example, the identity operator on $V$ has infinitely many square roots if $\dim V > 1$.

# *Linear Algebra Done Right*, by Sheldon Axler