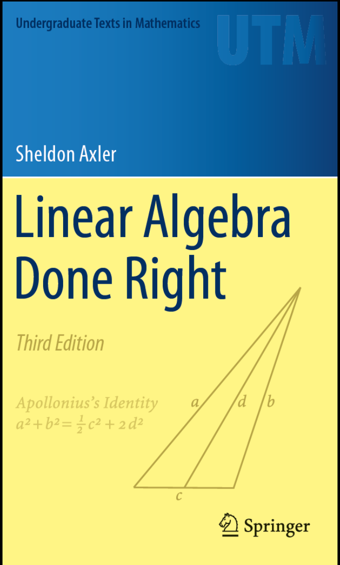
Polar Decomposition and SVD, part 1: Polar Decomposition



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional inner product space over $\mathbf{F}$.

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An operator $T \in \mathcal{L}(V)$ is called *positive* if T is self-adjoint and

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Characterization of positive operators

Let $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) T is positive;

(b) T is self-adjoint and all eigenvalues of T are nonnegative;

(c) T has a positive square root;

(d) T has a self-adjoint square root;

(e) there exists $R \in \mathcal{L}(V)$ such that $T = R^*R$.

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Notation: $\sqrt{T}$

If T is a positive operator, then $\sqrt{T}$ denotes the unique positive square root of T.

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Polar decomposition on $\mathbf{C}$:

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$\lvert z\rvert = 1$ $(\bar{z}z = 1)$	T is an isometry $(T^*T = I)$
$z = \left(\frac{z}{\lvert z\rvert}\right)\sqrt{\bar{z}z}$	$T = S\sqrt{T^*T}$ for some isometry S

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$$\|Tv\| = \|\sqrt{T^*T}v\|.$$

Define $S_1 : \operatorname{range}\sqrt{T^*T} \to \operatorname{range} T$ by

$$S_1(\sqrt{T^*T}v) = Tv.$$

The equation $\|\sqrt{T^*T}v\| = \|Tv\|$ shows that

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Extend S_1 to an isometry $S \in \mathcal{L}(V)$. Do this by defining S to be an isometry from $(\operatorname{range}\sqrt{T^*T})^\perp$ to $(\operatorname{range} T)^\perp$, then extend by linearity. ∎

Linear Algebra Done Right, by Sheldon Axler