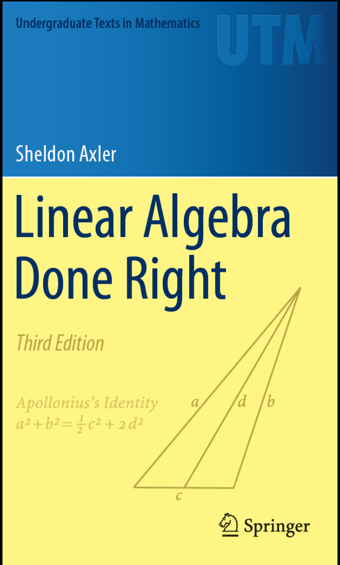
Orthogonal Complements



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes an inner product space over $\mathbf{F}$.

Orthogonal Complements

Definition: *orthogonal complement,* $U^\perp$

If U is a subset of V, then the *orthogonal complement* of U, denoted $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

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- If U is a subset of V, then $U \cap U^\perp \subset \{0\}$.
- If U and W are subsets of V and $U \subset W$, then $W^\perp \subset U^\perp$.

Direct Sum

Direct sum of a subspace and its orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

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$$v = u + (v - u),$$

showing that $v \in U + U^{\perp}$. Thus $V = U + U^{\perp}$.

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The orthogonal complement of the orthogonal complement

Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}.$$

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Definition: *orthogonal projection*, P_U

Suppose U is a finite-dimensional subspace of V. The *orthogonal projection* of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows:

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- $v - P_U v \in U^\perp$ for every $v \in V$;
- ${P_U}^2 = P_U$;
- $\|P_U v\| \leq \|v\|$ for every $v \in V$.

Linear Algebra Done Right, by Sheldon Axler