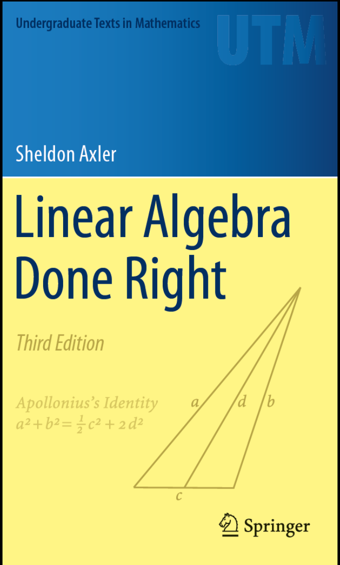
Null Spaces of Powers of an Operator



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Sequence of Null Spaces

Sequence of increasing null spaces

Suppose $T \in \mathcal{L}(V)$. Then

$$\{0\} = \operatorname{null} T^0 \subset \operatorname{null} T^1 \subset \cdots \subset \operatorname{null} T^k \subset \operatorname{null} T^{k+1} \subset \cdots .$$

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Thus $v \in \operatorname{null} T^{k+1}$. Hence $\operatorname{null} T^k \subset \operatorname{null} T^{k+1}$, as desired. ■

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which means that $v \in \operatorname{null} T^{m+k}$. This implies that $\operatorname{null} T^{m+k+1} \subset \operatorname{null} T^{m+k}$, completing the proof. ∎

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Null spaces stop growing

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At each of the strict inclusions in the chain above, the dimension increases by at least 1. Thus $\dim \operatorname{null} T^{n+1} \geq n + 1$, a contradiction because a subspace of V cannot have a larger dimension than n. ■

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$$\begin{aligned} &\dim(\operatorname{null} T^n \oplus \operatorname{range} T^n) \\ &\qquad = \dim \operatorname{null} T^n + \dim \operatorname{range} T^n \\ &\qquad = \dim V. \end{aligned}$$

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$$= 0.$$

Thus $T^n u = 0$. Thus $v = T^n u = 0$, showing that

$$(\operatorname{null} T^n) \cap (\operatorname{range} T^n) = \{0\}.$$

Thus $\operatorname{null} T^n + \operatorname{range} T^n$ is a direct sum. Now

$$\begin{aligned}\dim(\operatorname{null} T^n \oplus \operatorname{range} T^n) &= \dim \operatorname{null} T^n + \dim \operatorname{range} T^n \\ &= \dim V.\end{aligned}$$

The equation above implies that $\operatorname{null} T^n \oplus \operatorname{range} T^n = V$, as desired. ∎

Linear Algebra Done Right, by Sheldon Axler