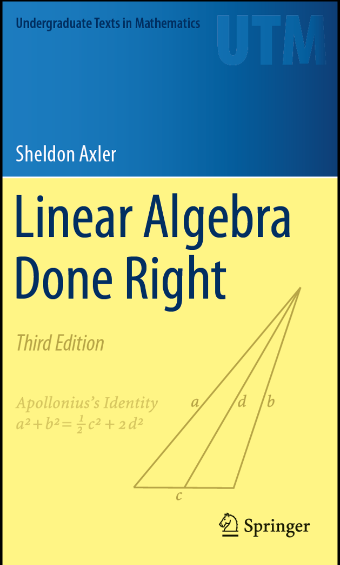
Minimization Problems



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Given a subspace U of V and a point $v \in V$, find a point $u \in U$ such that $\|v - u\|$ is as small as possible.

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Minimizing the distance to a subspace

Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

$$\|v - P_U v\| \leq \|v - u\|.$$

for all $u \in U$.

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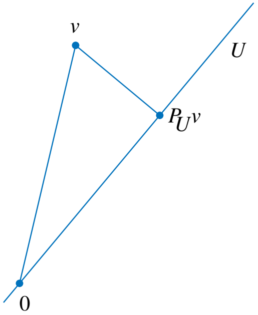

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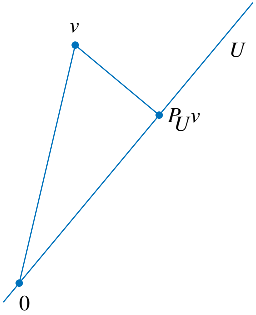

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Proof Suppose $u \in U$. Then

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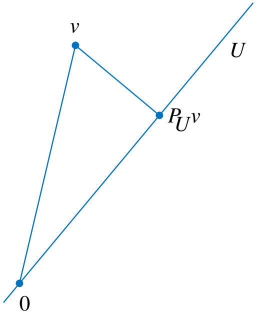

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$$\|v - P_U v\|^2 \leq \|v - P_U v\|^2 + \|P_U v - u\|^2$$
$$= \|(v - P_U v) + (P_U v - u)\|^2$$

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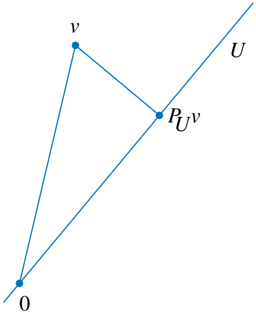

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$$\begin{aligned} \|v - P_U v\|^2 &\leq \|v - P_U v\|^2 + \|P_U v - u\|^2 \\ &= \|(v - P_U v) + (P_U v - u)\|^2 \\ &= \|v - u\|^2. \end{aligned}$$

Example Minimization Problem

Find $u \in \mathcal{P}_5(\mathbf{R})$ that approximates $\sin x$ as well as possible on the interval $[-\pi, \pi]$, in the sense that

$$\int_{-\pi}^{\pi} |\sin x - u(x)|^2 \, dx$$

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Here $V = C_{\mathbf{R}}[-\pi, \pi]$ with inner product $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)\, dx$. Also, $v \in C_{\mathbf{R}}[-\pi, \pi]$ is given by $v(x) = \sin x$, and $U = \mathcal{P}_5(\mathbf{R})$.

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$$\begin{aligned} u &= P_U v \\ &= \sum_{j=1}^{6} \langle v, e_j \rangle e_j, \end{aligned}$$

where $e_1, \ldots, e_6$ is an orthonormal basis of U.

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Then use the last equation to get

$$\begin{aligned} u = &\frac{105(1485 - 153\pi^2 + \pi^4)}{8\pi^6} x \\ &- \frac{315(1155 - 125\pi^2 + \pi^4)}{4\pi^8} x^3 \\ &+ \frac{693(945 - 105\pi^2 + \pi^4)}{8\pi^{10}} x^5 \end{aligned}$$

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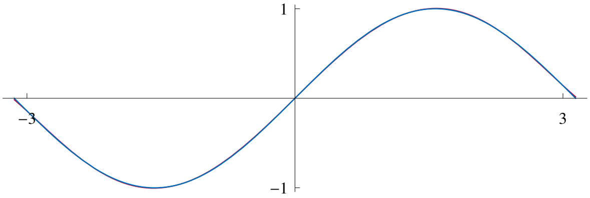

Graphs on $[-\pi, \pi]$ of $\sin x$ (blue) and its approximation $u(x)$ (red).

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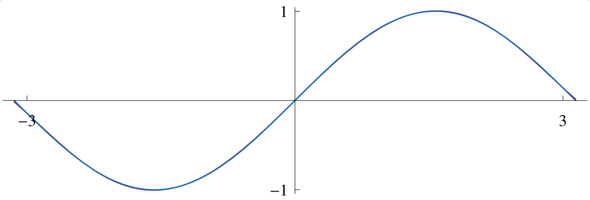

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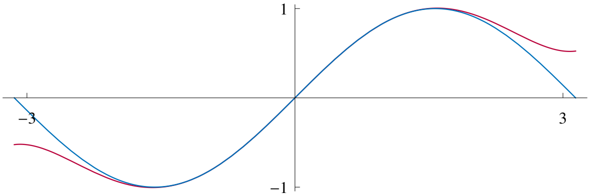

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Linear Algebra Done Right, by Sheldon Axler