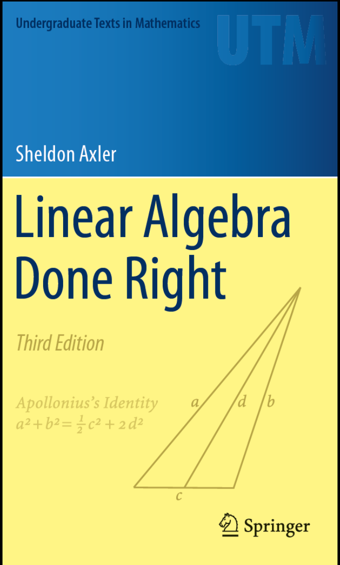
Matrices, part 1: The Matrix of a Linear Map



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

Definition and Notation for Matrix

Definition: *matrix*, $A_{j,k}$

Let m and n denote positive integers. An m-by-n *matrix* A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}.$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

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Example: Suppose $A = \begin{pmatrix} 8 & 4 & 5 \\ 1 & 9 & 7 \end{pmatrix}$. Then $A_{2,3} = 7$.

Matrix of a Linear Map

Definition: ***matrix of a linear map,*** $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The *matrix of* T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}\big(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)\big)$ is used.

Understanding the Matrix of a Linear Map

$$\mathcal{M}(T) = \begin{array}{c} \\ w_1 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & \dots & v_k & \dots & v_n \end{array} \\ \left(\begin{array}{ccccc} & & A_{1,k} & & \\ & & \vdots & & \\ & & A_{m,k} & & \end{array} \right) \end{array}.$$

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The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$:

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The k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$:

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The picture above should remind you that Tv_k can be computed from $\mathcal{M}(T)$ by multiplying each entry in the k^{th} column by the corresponding w_j from the left column, and then adding up the resulting vectors.

Examples of the Matrix of a Linear Map

Example: Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by

$$T(x, y) = (x + 3y, 2x + 5y, 7x + 9y).$$

Because $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$, the matrix of T with respect to the standard bases is the 3-by-2 matrix

$$\mathcal{M}(T) = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}.$$

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Example: Suppose $D \in \mathcal{L}\big(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R})\big)$ is the differentiation map defined by $Dp = p'$. Because $(x^n)' = nx^{n-1}$, the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbf{R})$ and $\mathcal{P}_2(\mathbf{R})$ is the 3-by-4 matrix

$$\mathcal{M}(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Addition of Matrices

Definition: *matrix addition*

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \dots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \dots & C_{m,n} \end{pmatrix}$$

$$= \begin{pmatrix} A_{1,1} + C_{1,1} & \dots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \dots & A_{m,n} + C_{m,n} \end{pmatrix}.$$

In other words, $(A + C)_{j,k} = A_{j,k} + C_{j,k}$.

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In the following result, the assumption is that the same bases are used for $\mathcal{M}(S + T)$, $\mathcal{M}(S)$, and $\mathcal{M}(T)$.

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$$= \begin{pmatrix} A_{1,1}+C_{1,1} & \dots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \dots & A_{m,n}+C_{m,n} \end{pmatrix}.$$

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The matrix of the sum of linear maps

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.

Scalar Multiplication of Matrices

Definition: ***scalar multiplication of a matrix***

The product of a scalar and a matrix is the matrix obtained by multiplying each entry in the matrix by the scalar:

$$\lambda \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{pmatrix}.$$

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In the following result, the assumption is that the same bases are used for $\mathcal{M}(\lambda T)$ and $\mathcal{M}(T)$.

The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

The Vector Space of Matrices

Notation: $\mathbf{F}^{m,n}$

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$\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn.

Linear Algebra Done Right, by Sheldon Axler