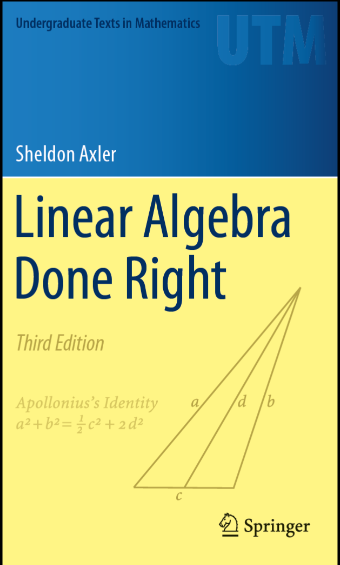
Positive Operators and Isometries, part 2: Isometries



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for all $v \in V$.
- In other words, an operator is an isometry if it preserves norms.

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$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$
and
$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2.$$

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$$\begin{aligned} Sv &= \langle v, e_1\rangle Se_1 + \cdots + \langle v, e_n\rangle Se_n \\ &= \lambda_1\langle v, e_1\rangle e_1 + \cdots + \lambda_n\langle v, e_n\rangle e_n. \end{aligned}$$

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Thus

$$\|Sv\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2.$$

Hence $\|v\| = \|Sv\|$. Thus S is an isometry.

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Clearly (c) implies (d).

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for $j, k = 1, \ldots, n$.

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for $j, k = 1, \ldots, n$. All vectors $u, v \in V$ can be written as linear combinations of $e_1, \ldots, e_n$, and thus the equation above implies that

$$\langle S^*Su, v \rangle = \langle u, v \rangle$$

for all $u, v \in V$.

Characterization of Isometries

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For $S \in \mathcal{L}(V)$, the following are equivalent:

(a) S is an isometry;

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(d) there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal;

(e) $S^*S = I$;

(f) $SS^* = I$;

(g) S^* is an isometry;

(h) S is invertible and $S^{-1} = S^*$.

Now suppose (d) holds.
Let $e_1, \ldots, e_n$ be an orthonormal basis of V such that $Se_1, \ldots, Se_n$ is orthonormal. Thus

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Now suppose (f) holds, so $SS^* = I$. If $v \in V$, then

$$\begin{aligned} \|S^*v\|^2 &= \langle S^*v, S^*v \rangle \\ &= \langle SS^*v, v \rangle \\ &= \langle v, v \rangle \\ &= \|v\|^2. \end{aligned}$$

Thus S^* is an isometry, showing that (g) holds.

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Now suppose (g) holds, so S^* is an isometry.

Using the implications (a) $\Rightarrow$ (e) and (a) $\Rightarrow$ (f) but with S replaced with S^* [and using the equation $(S^*)^* = S$], we conclude that

$$SS^* = I \quad \text{and} \quad S^*S = I.$$

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Thus S is an isometry, showing that (a) holds. ■

Description of isometries when $\mathbf{F} = \mathbf{C}$

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. The following are equivalent:

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Thus each eigenvalue of S has absolute value 1. Hence (b) holds.

We earlier showed that (b) implies (a). ■

Linear Algebra Done Right, by Sheldon Axler