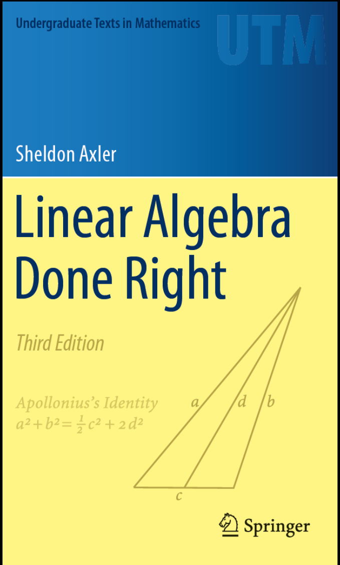
Invertibility and Isomorphic Vector Spaces



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

Invertible Linear Maps

Definition: *invertible*

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Inverse is unique

An invertible linear map has a unique inverse.

Invertibility

Notation: T^{-1}

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

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Invertibility is equivalent to injectivity and surjectivity

A linear map is invertible if and only if it is injective and surjective.

Dimension Determines Whether Vector Spaces Are Isomorphic

Definition: *isomorphism, isomorphic*

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- Two vector spaces are called *isomorphic* if there is an isomorphism from one vector space onto the other one.

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Dimension shows whether vector spaces are isomorphic

Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

The Dimension of $\mathcal{L}(V, W)$

$\mathcal{L}(V, W)$ *and* $\mathbf{F}^{m,n}$ *are isomorphic*

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$.

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$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W).$$

Matrix of a Vector

Definition: *matrix of a vector,* $\mathcal{M}(u)$

Suppose $u \in V$ and $v_1, \ldots, v_n$ is a basis of V. The *matrix of* u with respect to this basis is the n-by-1 matrix

$$\mathcal{M}(u) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

where $c_1, \ldots, c_n$ are the scalars such that

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The matrix of $(5, 8, 2)$ with respect to the standard basis of $\mathbf{F}^3$ is

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The matrix of $3 - 7x + 5x^2$ with respect to the standard basis of $\mathcal{P}_2(\mathbf{R})$ is

$$\begin{pmatrix} 3 \\ -7 \\ 5 \end{pmatrix}.$$

Linear Maps Act Like Matrix Multiplication

Linear maps act like matrix multiplication

Suppose $T \in \mathcal{L}(V, W)$ and $u \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tu) = \mathcal{M}(T)\mathcal{M}(u).$$

Operators

Definition: *operator*, $\mathcal{L}(V)$

- A linear map from a vector space to itself is called an *operator*.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

For $T \in \mathcal{L}(V)$ and $\dim V < \infty$, Injectivity $\iff$ Surjectivity

Injectivity is equivalent to surjectivity in finite dimensions

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- T is invertible;
- T is injective;
- T is surjective.

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or

T is injective $\Longleftrightarrow$ T is surjective.

Linear Algebra Done Right, by Sheldon Axler