Notation

- \( \mathbb{F} \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \).

- \( V \) denotes a vector space over \( \mathbb{F} \).
Dot Product on $\mathbb{R}^n$

This vector $x$ has length
$$\sqrt{x_1^2 + x_2^2}.$$
For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define the norm of $x$ by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. 

This vector $x$ has length $\sqrt{x_1^2 + x_2^2}$. 

Definition: Dot product

For $x, y \in \mathbb{R}^n$, the dot product of $x$ and $y$, denoted $x \cdot y$, is defined by $x \cdot y = x_1 y_1 + \cdots + x_n y_n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. 

$x \cdot x = \|x\|_2$ for all $x \in \mathbb{R}^n$. 

The dot product on $\mathbb{R}^n$ has the following properties:

1. $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$;
2. $x \cdot x = 0$ if and only if $x = 0$;
3. For $y \in \mathbb{R}^n$ fixed, the map from $\mathbb{R}^n$ to $\mathbb{R}$ that sends $x \in \mathbb{R}^n$ to $x \cdot y$ is linear;
4. $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}^n$. 

The dot product on $\mathbb{R}^2$ is the usual inner product, and it can be visualized geometrically as the projection of $x$ onto $y$, scaled by the length of $x$. 

Diagram: 

[Diagram showing a vector $x$ and its dot product with another vector $(x_1, x_2)$]
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\[ x \cdot y = x_1y_1 + \cdots + x_ny_n, \]

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. 

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- for $\mathbf{y} \in \mathbb{R}^n$ fixed, the map from $\mathbb{R}^n$ to $\mathbb{R}$ that sends $\mathbf{x} \in \mathbb{R}^n$ to $\mathbf{x} \cdot \mathbf{y}$ is linear;
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Recall that if $\lambda = a + bi$, where $a, b \in \mathbb{R}$, then

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For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define the norm of $z$ by $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$.

This suggests that the inner product of $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ with $z$ should equal $w_1 z_1 + \cdots + w_n z_n$. 
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For \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \), define the norm of \( z \) by

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Definition: inner product

An inner product on $V$ is a function that takes each ordered pair $(u, v)$ of elements of $V$ to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties:

- **Positivity**: $\langle v, v \rangle \geq 0$ for all $v \in V$;
- **Definiteness**: $\langle v, v \rangle = 0$ if and only if $v = 0$;
- **Additivity in first slot**: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$;
- **Homogeneity in first slot**: $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$;
- **Conjugate Symmetry**: $\langle u, v \rangle = \langle v, u \rangle^*$ for all $u, v \in V$. 
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  $$\langle v, v \rangle = 0 \text{ if and only if } v = 0;$$

- additivity in first slot
  $$\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle \text{ for all } u, v, w \in V;$$

- homogeneity in first slot
  $$\langle \lambda u, v \rangle = \lambda \langle u, v \rangle \text{ for all } \lambda \in F \text{ and all } u, v \in V;$$

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**Properties**

- **Inner Product Field (F)**: $\mathbb{R}$ or $\mathbb{C}$
- **Inner Product Space (V)**: Real or complex vector space
The *Euclidean inner product* on $\mathbb{F}^n$ is defined by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1\overline{z_1} + \cdots + w_n\overline{z_n}.$$
Examples of Inner Products

- The *Euclidean inner product* on $\mathbb{F}^n$ is defined by
  $$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$  

- If $c_1, \ldots, c_n$ are positive numbers, then an inner product can be defined on $\mathbb{F}^n$ by
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- An inner product can be defined on the vector space of continuous real-valued functions on the interval $[-1, 1]$ by
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- An inner product can be defined on $\mathcal{P}(\mathbb{R})$ by

$$\langle p, q \rangle = \int_{0}^{\infty} p(x) q(x) e^{-x} \, dx.$$
Definition: *inner product space*

An *inner product space* is a vector space $V$ along with an inner product on $V$. 

Properties of an inner product:

- For each fixed $u \in V$, the function $v \mapsto \langle v, u \rangle$ is a linear map from $V$ to $F$.
- $\langle 0, u \rangle = 0$ for every $u \in V$.
- $\langle u, 0 \rangle = 0$ for every $u \in V$.
- $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
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**Notation: $V$**

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