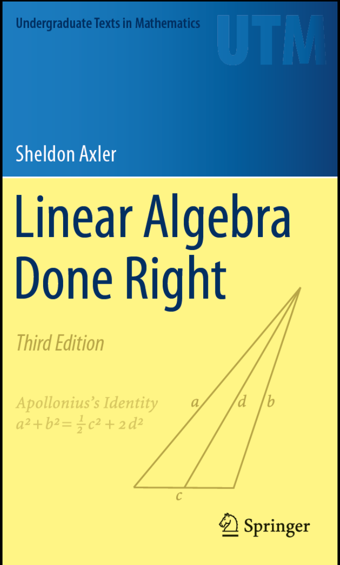
Generalized Eigenvectors



Eigenvalue and Eigenvector Review

Definition: *eigenvalue*

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an *eigenvalue* of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

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Definition: *eigenvector*

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$ is an eigenvalue of T. A vector $v \in V$ is called an *eigenvector* of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

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Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \operatorname{null}(T - \lambda I).$$

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Clearly

$$v \in \operatorname{null}(T - \lambda I) \Longleftrightarrow Tv = \lambda v.$$

Thus $E(\lambda, T)$ is the set of eigenvectors of T corresponding to λ, along with the 0 vector.

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Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. A vector $v \in V$ is called a *generalized eigenvector* of T corresponding to λ if $v \neq 0$ and

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Description of generalized eigenspaces

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Then

$$G(\lambda, T) = \operatorname{null}(T - \lambda I)^{\dim V}.$$

Example of Generalized Eigenspaces

Define $T \in \mathcal{L}(\mathbf{C}^3)$ by

$$T(z_1, z_2, z_3) = (4z_2, 0, 5z_3).$$

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The corresponding eigenspaces are easily seen to be

$$E(0, T) = \{(z_1, 0, 0) : z_1 \in \mathbf{C}\}$$

and

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We have

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Now

$$\mathbf{C}^3 = G(0, T) \oplus G(5, T).$$

Linearly Independent Generalized Eigenvectors

Linearly independent generalized eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T and $v_1, \ldots, v_m$ are corresponding generalized eigenvectors. Then $v_1, \ldots, v_m$ is linearly independent.

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The equation above implies that $a_1 = 0$. In a similar fashion, $a_j = 0$ for each j. Thus $v_1, \ldots, v_m$ is linearly independent. ∎

Linear Algebra Done Right, by Sheldon Axler