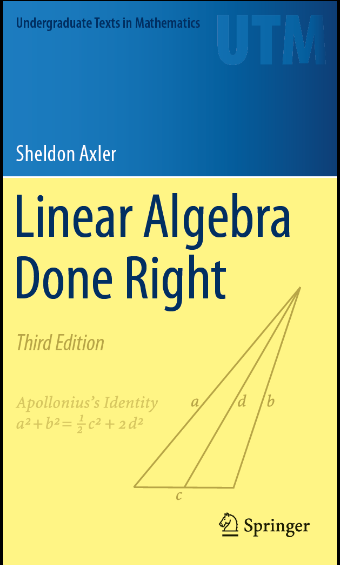
Existence of Eigenvalues



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- $\mathcal{L}(V) = \mathcal{L}(V, V)$

Powers of an Operator

Definition: T^m

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

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If $T \in \mathcal{L}(V)$, then
$$T^m T^n = T^{m+n} \quad \text{and} \quad (T^m)^n = T^{mn},$$
where m and n are allowed to be arbitrary integers if T is invertible and nonnegative integers if T is not invertible.

Polynomials Applied to an Operator

Definition: $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

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for $z \in \mathbf{F}$. Then $p(T)$ is the operator defined by

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Example: Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is the differentiation operator defined by $Dq = q'$ and p is the polynomial defined by

$$p(x) = 7 - 3x + 5x^2.$$

Then

$$p(D) = 7I - 3D + 5D^2;$$

thus

$$\big(p(D)\big)q = 7q - 3q' + 5q''$$

for every $q \in \mathcal{P}(\mathbf{R})$.

Algebraic Properties of $p(T)$

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This result is false on infinite-dimensional complex vector spaces.
Example: Define $T \in \mathcal{L}(\mathcal{P}(\mathbf{C}))$ by

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$$0 = a_0v + a_1Tv + \cdots + a_nT^nv.$$

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Thus $T - \lambda_jI$ is not injective for at least one j. Hence T has an eigenvalue. ∎

Linear Algebra Done Right, by Sheldon Axler