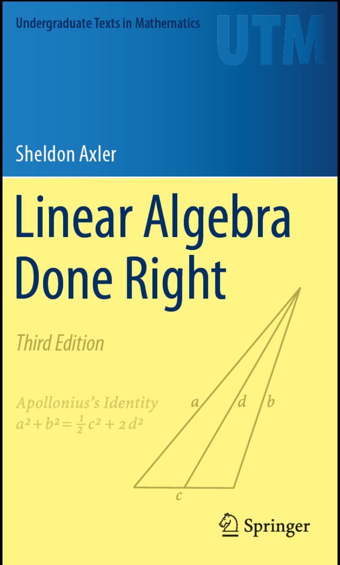
Duality, part 1: Dual Bases and Dual Maps



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

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- Define $\varphi\colon \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by $\varphi(p) = 3p''(5) + 7p(4)$. Then φ is a linear functional on $\mathcal{P}(\mathbf{R})$.

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- Define $\varphi\colon \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by $\varphi(p) = \int_0^1 p(x)\, dx$. Then φ is a linear functional on $\mathcal{P}(\mathbf{R})$.

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Dual basis is a basis of the dual space

Suppose $\dim V < \infty$. Then the dual basis of a basis of V is a basis of V'.

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- $(ST)' = T'S'$ for all $T \in \mathcal{L}(U, V)$ and all $S \in \mathcal{L}(V, W)$.

Linear Algebra Done Right, by Sheldon Axler