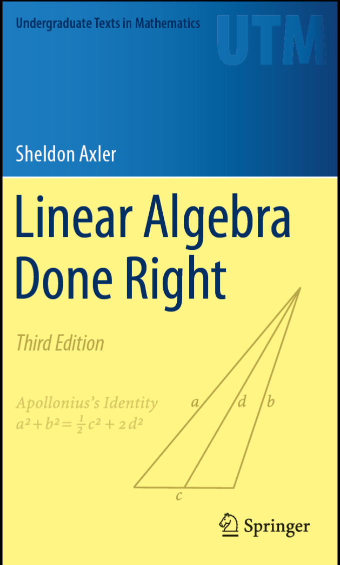
Complexification



Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- V denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

Complexification of a Vector Space

Definition: *complexification of* V, $V_{\mathbf{C}}$

Suppose V is a real vector space.

- The *complexification* of V, denoted $V_{\mathbf{C}}$, equals $V \times V$. An element of $V_{\mathbf{C}}$ is an ordered pair (u, v), where $u, v \in V$, but we will write this as $u + iv$.
- Addition on $V_{\mathbf{C}}$ is defined by
  $$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
  for $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on $V_{\mathbf{C}}$ is defined by
  $$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
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Basis of V is basis of $V_{\mathbf{C}}$

Suppose V is a real vector space.

- If $v_1, \ldots, v_n$ is a basis of V (as a real vector space), then $v_1, \ldots, v_n$ is a basis of $V_{\mathbf{C}}$ (as a complex vector space).
- The dimension of $V_{\mathbf{C}}$ (as a complex vector space) equals the dimension of V (as a real vector space).

Complexification of an Operator

Definition: *complexification of T, $T_{\mathbf{C}}$*

Suppose V is a real vector space and $T \in \mathcal{L}(V)$. The *complexification* of T, denoted $T_{\mathbf{C}}$, is the operator $T_{\mathbf{C}} \in \mathcal{L}(V_{\mathbf{C}})$ defined by

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Example: Suppose A is an n-by-n matrix of real numbers. Define $T \in \mathcal{L}(\mathbf{R}^n)$ by

$$Tx = Ax,$$

where elements of $\mathbf{R}^n$ are thought of as n-by-1 column vectors.

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Identifying the complexification of $\mathbf{R}^n$ with $\mathbf{C}^n$, we then have

$$T_{\mathbf{C}} z = Az$$

for each $z \in \mathbf{C}^n$, where again elements of $\mathbf{C}^n$ are thought of as n-by-1 column vectors.

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Matrix of $T_{\mathbf{C}}$ equals matrix of T

Suppose V is a real vector space with basis $v_1, \ldots, v_n$ and $T \in \mathcal{L}(V)$. Then $\mathcal{M}(T) = \mathcal{M}(T_{\mathbf{C}})$, where both matrices are with respect to the basis $v_1, \ldots, v_n$.

Invariant Subspaces

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The equations above show that U is invariant under T, completing the proof. ∎

Minimal Polynomial of the Complexification

Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Repeated application of the definition of $T_{\mathbf{C}}$ shows that

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Minimal polynomial of $T_{\mathbf{C}}$ equals minimal polynomial of T

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Eigenvalues of the Complexification

Real eigenvalues of $T_{\mathbf{C}}$

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Nonreal eigenvalues of $T_{\mathbf{C}}$ ***come in pairs***

Suppose V is a real vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Then λ is an eigenvalue of $T_{\mathbf{C}}$ if and only if $\bar{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

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Multiplicity of λ equals multiplicity of $\bar{\lambda}$

Suppose V is a real vector space, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$ is an eigenvalue of $T_{\mathbf{C}}$. Then the multiplicity of λ as an eigenvalue of $T_{\mathbf{C}}$ equals the multiplicity of $\bar{\lambda}$ as an eigenvalue of $T_{\mathbf{C}}$.

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Every real eigenvalue of $T_{\mathbf{C}}$ is also an eigenvalue of T, giving the desired result. ∎

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Multiplying together these two factors, we have

$$\begin{aligned}(z-\lambda)^m(z-\bar{\lambda})^m &= \big((z-\lambda)(z-\bar{\lambda})\big)^m \\ &= \big(z^2 - 2(\operatorname{Re}\lambda)z + |\lambda|^2\big)^m,\end{aligned}$$

which has real coefficients.

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Hence the characteristic polynomial of T is also

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Properties of the Characteristic Polynomial

Degree and zeros of characteristic polynomial

Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Then

- the coefficients of the characteristic polynomial of T are all real;
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Characteristic polynomial is a multiple of minimal polynomial

Suppose $T \in \mathcal{L}(V)$. Then

- the degree of the minimal polynomial of T is at most $\dim V$;
- the characteristic polynomial of T is a polynomial multiple of the minimal polynomial of T.

Linear Algebra Done Right, by Sheldon Axler