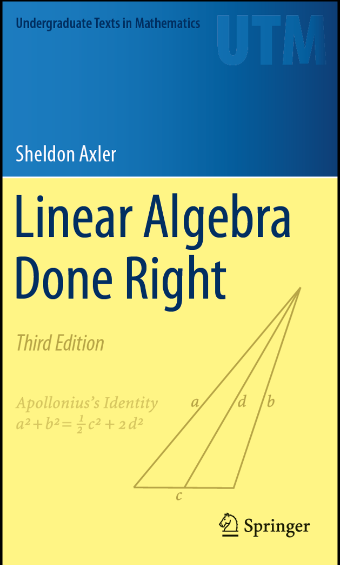
# The Characteristic Polynomial



# Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- $V$ denotes a finite-dimensional nonzero vector space over $\mathbf{F}$.

# Definition of Characteristic Polynomial

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Suppose $V$ is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of $T$, with multiplicities $d_1, \ldots, d_m$. The polynomial

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$$T(z_1, z_2, z_3) = (6z_1 + 3z_2 + 4z_3, 6z_2 + 2z_3, 7z_3).$$

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Because the eigenvalues of $T$ are $6$, with multiplicity $2$, and $7$, with multiplicity $1$, the characteristic polynomial of $T$ is

$$(z - 6)^2 (z - 7).$$

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- the characteristic polynomial of $T$ has degree $\dim V$;
- the zeros of the characteristic polynomial of $T$ are the eigenvalues of $T$.

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Because $(T - \lambda_j I)^{d_j}|_{G(\lambda_j, T)} = 0$, we conclude that $q(T)|_{G(\lambda_j, T)} = 0$, as desired. ■

*Linear Algebra Done Right*, by Sheldon Axler