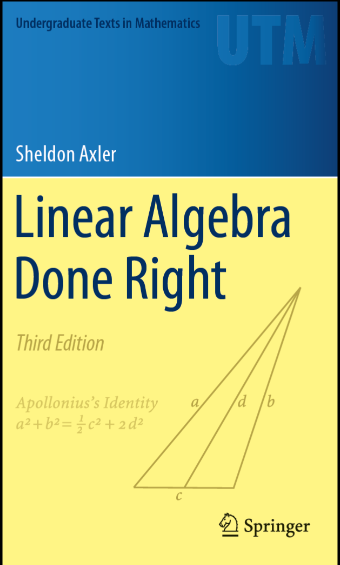
# Self-Adjoint and Normal Operators, Part 1: Adjoints



# Notation

- $\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.
- $V$ and $W$ denote finite-dimensional inner product spaces over $\mathbf{F}$.

# Adjoints

**Definition: *adjoint*, $T^*$**

Suppose $T \in \mathcal{L}(V, W)$. The *adjoint* of $T$ is the function $T^*: W \to V$ such that

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- $I^* = I$, where $I$ is identity operator on $V$;
- $(ST)^* = T^* S^*$ for all $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$ (here $U$ is an inner product space over $\mathbf{F}$).

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Thus $(ST)^*u = T^*(S^*u)$, as desired. ∎

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Replacing $T$ with $T^*$ in (a) gives (c).

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Finally, replacing $T$ with $T^*$ in (d) gives (b). ∎

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### *Condition for Surjectivity*

Suppose $T \in \mathcal{L}(V, W)$. Then $T$ is surjective if and only if $T^*$ is injective.

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The *conjugate transpose* of an $m$-by-$n$ matrix is the $n$-by-$m$ matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

Example: The conjugate transpose of the matrix $\begin{pmatrix} 2 & 3+4i & 7 \\ 6 & 5 & 8i \end{pmatrix}$ is the matrix

$$\begin{pmatrix} 2 & 6 \\ 3-4i & 5 \\ 7 & -8i \end{pmatrix}.$$

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**Definition: *conjugate transpose***

The *conjugate transpose* of an $m$-by-$n$ matrix is the $n$-by-$m$ matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

Example: The conjugate transpose of the matrix $\begin{pmatrix} 2 & 3+4i & 7 \\ 6 & 5 & 8i \end{pmatrix}$ is the matrix

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***The matrix of $T^*$***

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of $V$ and $f_1, \ldots, f_m$ is an orthonormal basis of $W$. Then

$$\mathcal{M}\big(T^*, (f_1, \ldots, f_m), (e_1, \ldots, e_n)\big)$$

is the conjugate transpose of

$$\mathcal{M}\big(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m)\big).$$

# *Linear Algebra Done Right*, by Sheldon Axler